HW-1 Solutions

1)

3.1)

$$\text{maximize} \quad -2x_1 + 3x_2 - 5x_3 - x_4$$

subject to

$$x_1 - 3x_2 + x_3 - 2x_4 - x_5 = 12$$
$$5x_1 + x_2 + 4x_3 - x_4 - x_6 = 10$$
$$-3x_1 + 2x_2 - x_3 + x_4 = 8$$
$$x_1, x_2, x_3, x_4, x_6 \geq 0$$

3.2)

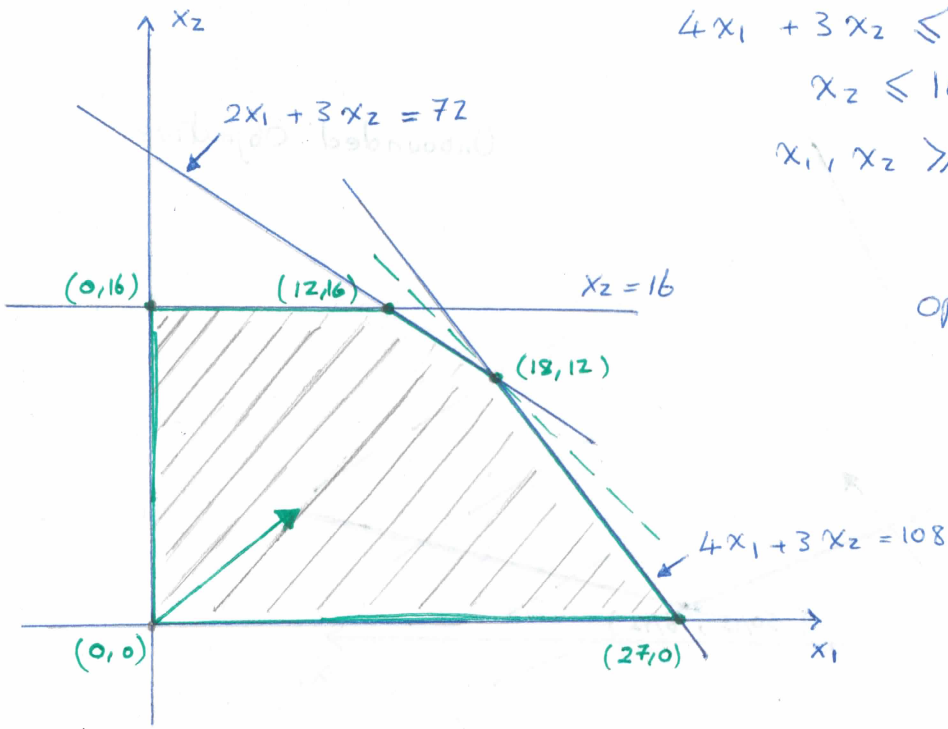
Let: $x_1 \equiv$ # of Chair A produced per week

$x_2 \equiv$ # of Chair B produced per week

$$\text{maximize} \quad z = 10x_1 + 9x_2$$

subject to

$$2x_1 + 3x_2 \leq 72$$
$$4x_1 + 3x_2 \leq 108$$
$$x_2 \leq 16$$
$$x_1, x_2 \geq 0$$

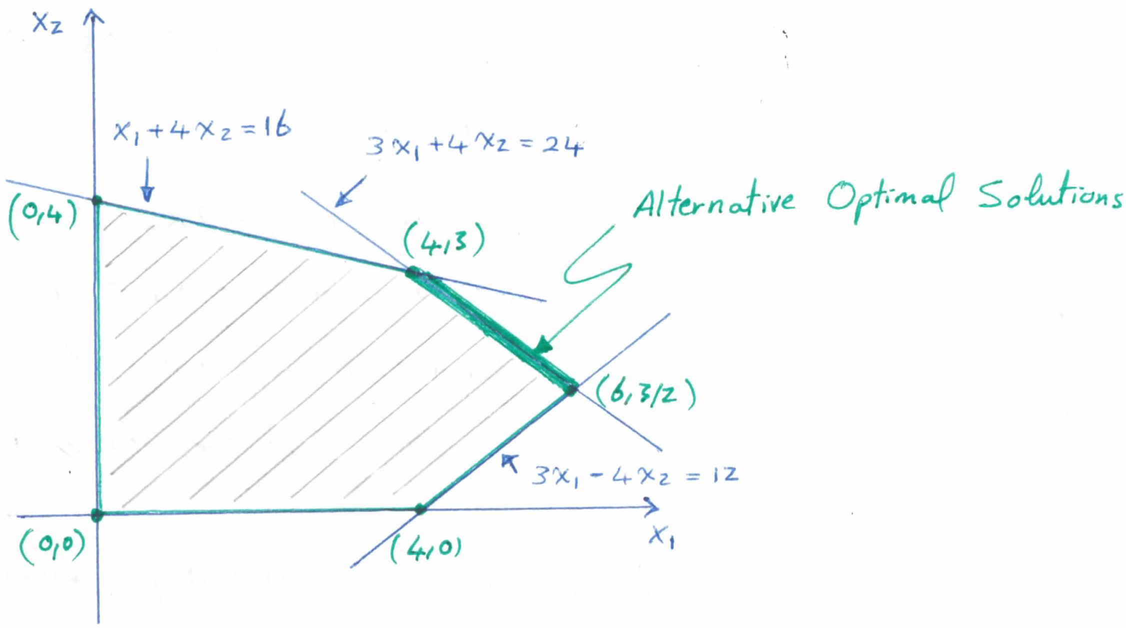


optimal solution is

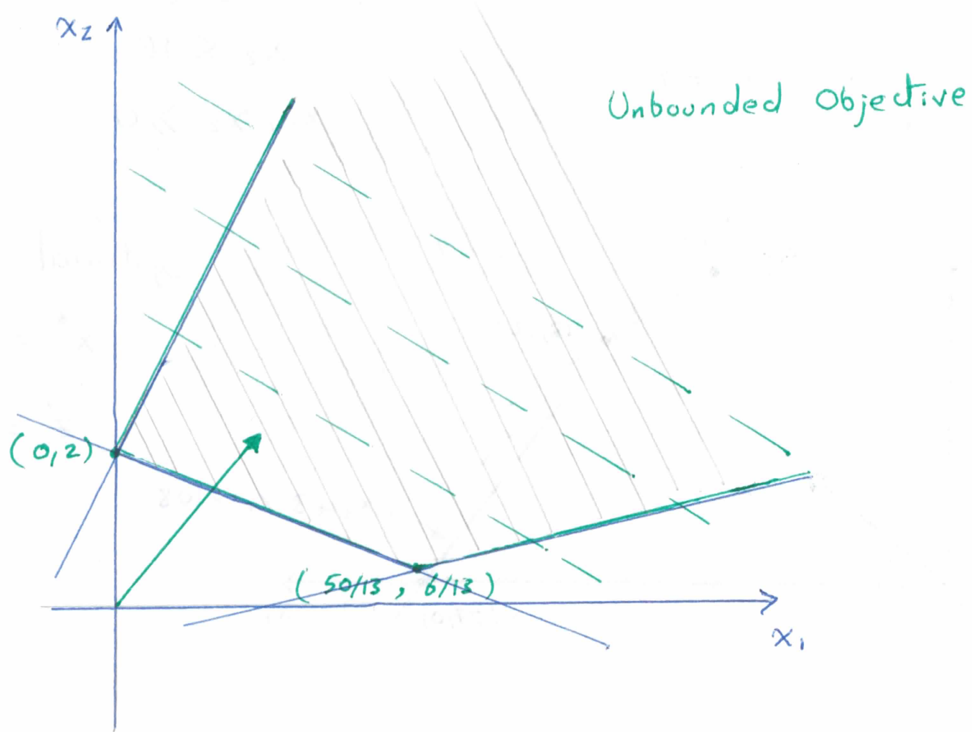
$x^* = (18, 12)$

$z^* = 288$

3.5) There are alternative optimal solutions. The optimal objective value is $z^* = 48$, and the optimal solution set is given by: $\left\{ x : x = \lambda \begin{bmatrix} 4 \\ 3 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 6 \\ 3/2 \end{bmatrix}, \ \lambda \in [0,1] \right\}$.



3.6) There is no finite optimal solution, that is, the value of the objective function can be made arbitrarily large.

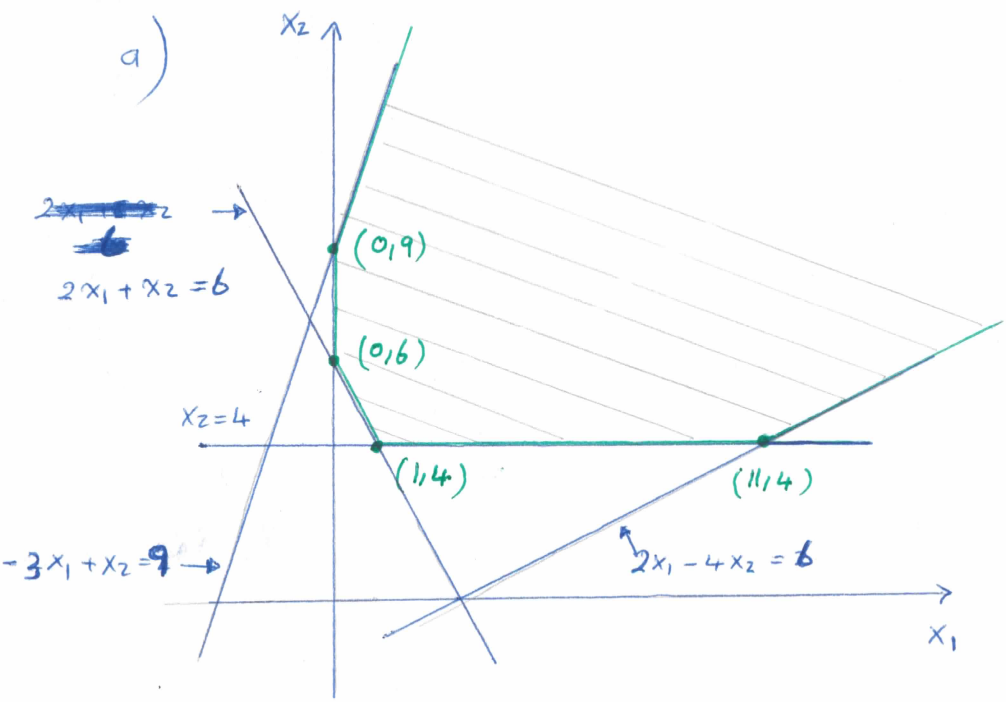


3.11)

Nonbasic Variables	Basic Variables	Basis Matrix B
$x_1 = x_2 = 0$	$x_3 = -5$, $x_4 = 25$	$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$
$x_1 = x_3 = 0$	$x_2 = 15/2$, $x_4 = 15/2$	$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$
$x_1 = x_4 = 0$	$x_2 = 75/7$, $x_3 = 15/7$	$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$
$x_2 = x_3 = 0$	$x_1 = -15$, $x_4 = 45$	$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$
$x_2 = x_4 = 0$	$x_1 = 75/4$, $x_3 = -45/4$	$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$
$x_3 = x_4 = 0$	$x_1 = 3$, $x_2 = 9$	$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

3.14)

a)



b) Extreme points:

(0,9) , (0,6)

(1,4) , (11,4)

3.15)

$S' \subseteq S$ and $z' \leq z^*$

3.17)

The feasible region is nonempty and the extreme directions of the feasible region are:

$$d_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } d_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, a finite optimal solution exists if

$$cd_1 = (c_1, c_2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_2 \leq 0$$

$$cd_2 = (c_1, c_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2c_1 + c_2 \leq 0$$

3)

3.1)

maximize $-2x_1 + 3x_2 - 5x_3 - x_4$

subject to

$$x_1 - 3x_2 + x_3 - 2x_4 - x_5 = 12$$

$$5x_1 + x_2 + 4x_3 - x_4 - x_6 = 10$$

$$-3x_1 + 2x_2 - x_3 + x_4 = 8$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

No basic feasible solution can be found for this problem. Therefore, it is infeasible

Optimality condition: a solution is optimal

3.2) if $dz/dx_j \leq 0$ for all non-basic x_j

in standard form

$$\text{maximize} \quad z = 10x_1 + 9x_2$$

subject to

$$2x_1 + 3x_2 + x_3 = 72$$
$$4x_1 + 3x_2 + x_4 = 108$$
$$x_2 + x_5 = 16$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

for the optimal solution

$$x_1^* = 18 \qquad x_2^* = 12$$

Then

$$x_3 = 0 \quad , \quad x_4 = 0 \quad , \quad x_5 = 4$$

Therefore:

non-basic variables $x_3 = x_4 = 0$

basic variables $x_1 = 18$, $x_2 = 12$, $x_5 = 4$

$$C = [10 \quad 9 \quad 0 \quad 0 \quad 0]^T$$

$$C_B = [10 \quad 9 \quad 0]^T$$

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 4 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{dz}{dx_3} = -(C_B^T B^{-1} a_3 - c_3) = -1 < 0$$

$$\frac{dz}{dx_4} = -(C_B^T B^{-1} a_4 - c_4) = -2 < 0$$

since both dz/dx_3 and $dz/dx_4 < 0$

then $x_1 = 18$ $x_2 = 12$ is optimal

4)

3.1)

minimize $z = 12y_1 + 10y_2 + 8y_3$

subject to

$$y_1 + 5y_2 - 3y_3 \geq -2$$
$$-3y_1 + y_2 + 2y_3 \geq 3$$
$$y_1 + 4y_2 - y_3 \geq -5$$
$$-2y_1 - y_2 + y_3 \geq -1$$
$$-y_1 \geq 0$$
$$-y_2 \geq 0$$
y_3 is unrestricted

This problem is unbounded.

3.2)

minimize $z = 72y_1 + 108y_2 + 16y_3$

subject to

$$2y_1 + 4y_2 \geq 10$$
$$3y_1 + 3y_2 + y_3 \geq 9$$
$$y_1, y_2, y_3 \geq 0$$

Optimal solution

$y_1^* = 1 \rightarrow 2x_1 + 3x_2 \leq 72$ binding

$y_2^* = 2 \rightarrow 4x_1 + 3x_2 \leq 108$ binding

$y_3^* = 0 \rightarrow x_2 \leq 16$ non-binding

$z^* = 288$

3.5)

minimize $z = 16y_1 + 24y_2 + 12y_3$

subject to

$$y_1 + 3y_2 + 3y_3 \geqslant 6$$

$$4y_1 + 4y_2 - 4y_3 \geqslant 8$$

$$y_1, y_2, y_3 \geqslant 0$$

Optimal solution

$y_1^* = 0 \rightarrow x_1 + 4x_2 \leqslant 16$ non-binding

$y_2^* = 2 \rightarrow 3x_1 + 4x_2 \leqslant 24$ binding

$y_3^* = 0 \rightarrow 3x_1 - 4x_2 \leqslant 12$ non-binding

3.6)

minimize $z = 2y_1 + 10y_2 + 2y_3$

subject to

$$-2y_1 + 2y_2 + y_3 \geqslant 1$$

$$y_1 + 5y_2 - 4y_3 \geqslant 2$$

$$y_1, y_3 \geqslant 0 \qquad y_2 \leqslant 0$$

This problem is infeasible.

3.14)

minimize $z = 6y_1 + 9y_2 + 4y_3 + 6y_4$

subject to

$$2y_1 - 3y_2 + 2y_4 \geqslant -7$$

$$y_1 + y_2 + y_3 - 4y_4 \geqslant 2$$

$$y_1, y_3 \leqslant 0 \qquad y_2, y_4 \geqslant 0$$

Optimal solution

$y_1^* = 0 \longrightarrow 2x_1 + x_2 \geqslant 6$ non-binding

$y_2^* = 2 \longrightarrow -3x_1 + x_2 \leqslant 9$ binding

$y_3^* = 0 \longrightarrow x_2 \geqslant 4$ non-binding

$y_4^* = 0 \longrightarrow 2x_1 - 4x_2 \leqslant 6$ non-binding

$z^* = 18$