EXERCISES

.1. Transform the following linear program into the standard form given by (3.33).

$$\text{minimize } z = 2x_1 - 3x_2 + 5x_3 + x_4$$

subject to

$$-x_1 + 3x_2 - x_3 + 2x_4 \leq -12$$
$$5x_1 + x_2 + 4x_3 - x_4 \geq 10$$
$$3x_1 - 2x_2 + x_3 - x_4 = -8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

.2. Modern Furniture, Inc., produces two types of wooden chairs. The manufacture of Chair A requires 2 hours of assembly time and 4 hours of finishing time. Chair B requires 3 hours to assemble and 3 hours to finish. Modern estimates that next week 72 hours will be available for assembly operations, and 108 hours will be available in the finishing shop. The unit profits for Chairs A and B are \$10 and \$9, respectively. If it is estimated that the maximum demand for Chair B is 16, what is the optimal product mix? Formulate a linear programming model and solve the resulting model graphically.

.3. Solve the following linear program graphically.

$$\text{minimize } z = 4x_1 + 5x_2$$

subject to

$$3x_1 + 2x_2 \leq 24$$
$$x_1 \geq 5$$
$$3x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

4. Solve the following linear program graphically.

$$\text{minimize } z = x_1 - 4x_2$$

subject to

$$x_1 + x_2 \leq 12$$
$$-2x_1 + x_2 \leq 4$$
$$x_2 \leq 8$$
$$x_1 - 3x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

5. Solve the following linear program graphically.

$$\text{maximize } z = 6x_1 + 8x_2$$

subject to

$$x_1 + 4x_2 \leq 16$$
$$3x_1 + 4x_2 \leq 24$$
$$3x_1 - 4x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

3.6. Solve the following linear program graphically.

$$\text{maximize } z = x_1 + 2x_2$$

subject to

$$-2x_1 + x_2 \leq 2$$
$$2x_1 + 5x_2 \geq 10$$
$$x_1 - 4x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

3.7. Show that the halfspace $H^- = \{\mathbf{x} : \mathbf{a}\mathbf{x} \leq \alpha\}$ is a convex set.

3.8. Let

$$\mathbf{a}_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Illustrate graphically the following:

(a) The set of all linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.
(b) The set of all nonnegative linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.
(c) The set of all convex combinations of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

3.9. Given the polyhedral set $S = \{(x_1, x_2) : x_1 + x_2 \leq 10, -x_1 + x_2 \leq 6, x_1 - 4x_2 \leq 0\}$.
(a) Find all extreme points of S.
(b) Represent the point $\mathbf{x} = (2,4)$ as a convex combination of the extreme points.

3.10. Let S_1 and S_2 be convex sets. Show that the set $S_1 \cap S_2$ is a convex set. Is this also true of $S_1 \cup S_2$?

3.11. Given the following system of linear equations.

$$x_1 + 3x_2 - x_3 + x_4 = 30$$
$$2x_1 + x_2 + 2x_3 + x_4 = 15$$

(a) Find all basic solutions.
(b) For each basic solution, specify the basic and nonbasic variables and the basis matrix $\mathbf{B}$.

3.12. Consider the following linear programming problem.

$$\text{maximize } z = 2x_1 + x_2$$

subject to

$$x_1 + 2x_2 \leq 20$$
$$-3x_1 + 4x_2 \leq 20$$

$$3x_1 + 2x_2 \leq 36$$

$$x_1, x_2 \geq 0$$

(a) Sketch the feasible region.
(b) Write the problem in standard equality form by adding slack variables.
(c) Identify the defining variable for each hyperplane bounding the feasible region, and specify the basic and nonbasic variables for each extreme point.
(d) Graphically determine the optimal extreme point and specify the optimal basis matrix.

3.13. Given the polyhedral set $S = \{(x_1, x_2) : x_1 + x_2 \geq 6, x_1 \geq 2, -2x_1 + x_2 \leq 4, x_1 - x_2 \leq 4, x_1, x_2 \geq 0\}$.
(a) Find all extreme points and extreme directions of S.
(b) Represent the point $\mathbf{x} = (5,8)$ as a convex combination of the extreme points and a nonnegative combination of the extreme directions.

3.14. Consider the following linear programming problem.

$$\text{maximize } z = -7x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \geq 6$$

$$-3x_1 + x_2 \leq 9$$

$$x_2 \geq 4$$

$$2x_1 - 4x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

(a) Sketch the feasible region and identify the optimal solution.
(b) Identify all extreme points and extreme directions.
(c) Reformulate the problem in terms of convex combinations of the extreme points and nonnegative combinations of the extreme directions as in (3.104–3.107). Solve the resulting problem and interpret the solution.
(d) Change the objective function to maximize $z = 4x_1 - x_2$ and repeat part (c).
(e) Discuss the practicality of using the procedure in parts (c) and (d) for large problems.

3.15. Let $S = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and let z^* be the optimal objective value associated with the linear program: maximize $z = \mathbf{cx}$ subject to $\mathbf{x} \in S$. Suppose that a constraint is added to the problem and results in the new feasible region S' with the corresponding optimal objective value z'. What is the relationship between S and S'? What is the relationship between z^* and z'?

3.16. Let $S = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and let z^* be the optimal objective value associated with the linear program: maximize $z = \mathbf{cx}$ subject to $\mathbf{x} \in S$. Suppose that a constraint is deleted from the problem and results in the new feasible region S' with the corresponding optimal objective value z'. What is the relationship between S and S'? What is the relationship between z^* and z'?

3.17. Mathematically characterize the set of objective coefficients (c_1, c_2) for which the following linear program has a finite optimal solution.

$$\text{maximize } z = c_1x_1 + c_2x_2$$

subject to

$$x_1 - 2x_2 \leq 8$$

$$x_1 \geq 2$$

$$2x_1 + 3x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

3.18. Consider the problem: (LP) maximize $\mathbf{cx}$ subject to $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Suppose that $\bar{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are both optimal solutions to (LP). Show that $\alpha\bar{\mathbf{x}} + (1 - \alpha)\hat{\mathbf{x}}$ is also optimal to (LP) for all $\alpha \in [0,1]$.

3.19. Let S be a nonempty polyhedral set defined by $S = \{\mathbf{x} \in E^n : \mathbf{Ax} \leq \mathbf{b}\}$. Assuming that the dimension of S is n, formulate a linear programming problem for determining the largest n-dimensional sphere that can be completely contained in S.

3.20. Let S be a nonempty convex subset of E^n. Show that $\bar{\mathbf{x}}$ is an extreme point of S if and only if the set $S - \{\bar{\mathbf{x}}\}$ is a convex set.