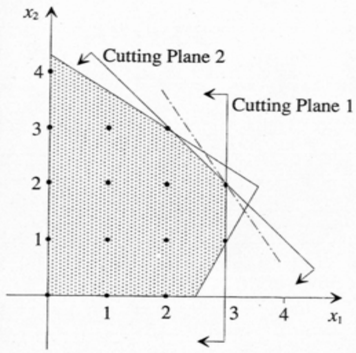

Figure 11.31 Cutting planes of Example 11.4.

However, the performance of the algorithm can be very slow in practice. For this reason, many enhancements and alternative cutting-plane techniques have been developed. The interested reader is referred to Taha (1975) and Salkin and Mathur (1989).

MMARY

In this chapter, we introduced integer linear programming models. After a brief discussion of computational complexity, several modeling techniques were presented for developing integer programming models. It was seen that integer programming problems are combinatorial in nature and are much more difficult to solve than linear programming problems in which the variables are continuous.

There are a number of methods for solving integer programming problems. However, exact solutions can usually be found only for moderate-size problems unless the problem exhibits some special characteristics. The basic solution methods presented in this chapter include branch-and-bound enumeration, implicit enumeration, and cutting-plane methods, as well as the graphical solution of simple integer programming problems.

EXERCISES

11.1. A company is planning the weekly production of three products, each of which requires two machining operations to produce. Each unit of Product A requires 12 minutes on Machine 1 and 9 minutes on Machine 2. Similarly, Product B needs 10 minutes on Machine 1 and 10 minutes on Machine 2, whereas the corresponding

values for Product C are 13 and 11. Machines 1 and 2 are each available for 40 hours during the week. If any units of Product A are manufactured a one-time setup and fixturing cost of \$10 and \$13 are incurred on Machines 1 and 2, respectively. The setup costs for Product B are \$11 on Machine 1 and \$9 on Machine 2. The corresponding costs for Product C are \$9 and \$15. The unit profits of Products A, B, and C are \$8, \$7, and \$9, respectively. Assuming that the company can sell all the units that it can produce, formulate an integer programming model to maximize profit.

11.2. A company is considering several investment opportunities, each of which differs in the initial capital required. The accounting department has done a thorough analysis of each of these investments and has estimated the long-term profit of each. The initial capital required and estimated profits (in millions of dollars) are summarized in Table 11.8. Investments 1 and 4 are considered high-risk investments and management has decided to invest in at most one of these. In addition, Investment 6 is contingent upon also investing in Investment 3. If \$120 million of initial capital is available, formulate an integer programming model to determine the optimal investment strategy.

TABLE 11.8

Investment	Initial capital	Estimated profit
1	26	18
2	34	12
3	18	7
4	45	24
5	31	11
6	39	15
7	23	9
8	13	6

11.3. **(a)** Show how integer programming techniques can be used to ensure that an integer variable x can only assume the values 3, 5, 13, or 21.
(b) Show how integer programming techniques can be used to reduce the following constraint to linear integer form.

$$|a_1x_1 + \cdots + a_2x_2| \geq b, \qquad \text{where } b > 0$$

11.4. Suppose that $-u_1 \leq f_1(\mathbf{x}) \leq u_1$ and $-u_2 \leq f_2(\mathbf{x}) \leq u_2$, where u_1 and u_2 are constants. Show how to use integer programming techniques to model the following.
(a) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$.
(b) If $f_1(\mathbf{x}) > 0$, then $f_2(\mathbf{x}) \geq 0$.
(c) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$, but both are not positive.

11.5. Consider the following constraint set. At least two of the following must hold:

$$x_1 + 2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 12$$

$$-x_1 + 3x_2 \leq 3$$

(a) Sketch this feasible region. Is the feasible region convex?
(b) Use integer programming techniques to write an equivalent linear integer model.

11.6. A city is trying to establish a municipal emergency ambulance service that can adequately service all sections of the city. The planning committee has established eight potential sites for the emergency service facilities. However, due to time and distance requirements, each potential site can provide coverage to only a subset of the city's six sections. Table 11.9 summarizes the sections for which each site provides coverage. Formulate an integer programming model for determining the fewest number of facilities that will provide all sections of the city with adequate coverage.

TABLE 11.9

Site	Sections of city covered
1	A, B, E
2	C, D
3	B, C, D
4	A, D, F
5	B, F
6	A, D, E, F
7	A, C, E
8	B, D, F

11.7. Formulate the following problem as a mixed integer programming problem.

maximize $z = f_1(x_1) + f_2(x_2)$

subject to:

at least two of the following hold: $x_1 + 3x_2 \leq 12$

$$2x_1 + x_2 \leq 16$$

$$x_1 + x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

where

$$f_1(x_1) = \begin{cases} 10 + 2x_1, & \text{if } 0 \leq x_1 \leq 5 \\ 15 + x_1, & \text{if } x_1 \geq 5 \end{cases}$$

$$f_2(x_2) = \begin{cases} 8 + x_2, & \text{if } 0 \leq x_2 \leq 2 \\ 4 + 3x_2, & \text{if } x_2 \geq 2 \end{cases}$$

11.8. A company manufactures two products, A and B, which are produced using two raw materials. Currently, both products are in high demand and it is anticipated that all that is manufactured can be sold. The selling price for one unit of Product A is \$3.20 and the selling price for Product B is \$2.70. Product A requires 4.2 pounds of Raw Material 1 and 1.7 pounds of Raw Material 2, and the corresponding figures for Product B are 3.9 pounds and 1.1 pounds. Raw Materials 1 and 2 are purchased from a local vendor and the purchase prices are determined according to the schedule shown in Table 11.10. Prior to sale, Products A and B must also undergo final inspection and packaging. Final inspection and packaging of each unit of Product A requires 1.2 hours, and Product B requires 0.9 hour per unit. If there are 480 hours available for final inspection and packaging, formulate an integer programming problem for determining the product mix that maximizes profit.

TABLE 11.10

Raw material 1		Raw material 2	
First 500 lb	\$0.30/lb	First 200 lb	\$0.40/lb
Next 1000 lb	\$0.25/lb	Next 600 lb	\$0.35/lb
Next 1000 lb	\$0.20/lb		
Maximum Available	2500 lb	Maximum Available	800 lb

11.9. ABC, Inc., is considering several investment options. Each option has a minimum investment required as well as a maximum investment allowed. These restrictions along with the expected return are summarized in the Table 11.11. (Figures are in millions of dollars.)

TABLE 11.11

Option	Minimum investment	Maximum investment	Expected return (%)
1	3	27	13
2	2	12	9
3	9	35	17
4	1	15	10
5	12	46	22
6	4	18	12

Because of the high-risk nature of Option 5, company policy requires that the total amount invested in Option 5 be no more than the combined amount invested in Options 2, 4, and 6. In addition, if an investment is made in Option 3, it is required that at least a minimum investment be made in Option 6. ABC has \$80 million to invest and obviously wants to maximize its total expected return on investment. Formulate an integer programming model to determine which options to invest in and how much should be invested.

11.10. Consider the following integer programming problem.

maximize $z = -3x_1 + 4x_2$

subject to:

$$6x_1 - 4x_2 \geq 15$$
$$x_1 + x_2 \geq 5$$
$$4x_1 + 2x_2 \leq 31$$
$$x_1, x_2 \geq 0$$
$$x_1, x_2 \text{ integers}$$

(a) Solve this problem graphically.
(b) Solve the LP relaxation graphically. Round the optimal solution to the LP relaxation in every possible way (i.e., round each noninteger value either up or down). Check these points for feasibility and compute the objective values of those that are feasible. Compare these with the solution found in part (a).
(c) Solve this problem by the branch-and-bound procedure described in Example 11.2. Solve the LP relaxation of each subproblem graphically.

1.11. Consider the following problem.

$$\text{maximize } z = 2x_1 + x_2$$

subject to:

$$2x_1 - 2x_2 \leq 3$$
$$-2x_1 + x_2 \leq 2$$
$$2x_1 + 2x_2 \leq 13$$
$$x_1, x_2 \geq 0$$
$$x_1, x_2 \text{ integers}$$

(a) Solve this problem graphically.
(b) Solve this problem by the branch-and-bound procedure described in Example 11.2. Solve the LP relaxation of each subproblem graphically.

1.12. Solve the following problem by the implicit enumeration procedure described in Example 11.3.

$$\text{minimize } z = 3x_1 + 2x_2 + 5x_3 + x_4$$

subject to:

$$-2x_1 + x_2 - x_3 - 2x_4 \leq -2$$
$$-x_1 - 5x_2 - 2x_3 + 3x_4 \leq -3$$
$$x_j \text{ binary, for all } j$$

1.13. Consider the following zero–one integer programming problem.

$$\text{maximize } z = -4x_1 + 3x_2 - 2x_3 + 7x_4$$

subject to:

$$-x_1 + 2x_2 + x_3 + 4x_4 \geq 6$$
$$-2x_1 + 2x_2 + x_3 + x_4 \leq 2$$
$$x_j \text{ binary, for all } j$$

(a) Transform this problem into the form minimize **cx** subject to **Ax** ≤ **b**, **x** binary, where **c** ≥ **0**.
(b) Solve the problem formulated in part (a) by implicit enumeration. Give the solution to the original problem.

11.14. Solve the following problem by implicit enumeration.

$$\text{minimize } z = 3x_1 + x_2 + 2x_3 - x_4$$

subject to:

$$2x_1 - 3x_2 - 4x_3 + 5x_4 \leq -1$$
$$x_1 + 2x_2 - x_3 + x_4 \geq 1$$
$$4x_1 + x_2 + x_3 + 2x_4 \geq 3$$
$$x_j \text{ binary, for all } j$$

11.15. Solve the following problem by implicit enumeration.

$$\text{minimize } z = 4x_1 + 5x_2 - 6x_3 + 2x_4 - 3x_5$$

subject to:

$$4x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 \leq 5$$
$$x_1 + 5x_2 - 2x_3 - 2x_4 - 2x_5 \geq 1$$
$$x_j \text{ binary, for all } j$$

11.16. Consider the following problem.

$$\text{maximize } z = 4x_1 + 8x_2$$

subject to:

$$2x_1 + 2x_2 \leq 19$$
$$-2x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$
$$x_1, x_2 \text{ integers}$$

(a) Solve this problem graphically.
(b) Solve this problem by the branch-and-bound procedure. Solve the LP relaxation of each subproblem graphically.
(c) Solve this problem by the cutting-plane method described in Example 11.4. Derive an expression for each cut in terms of x_1 and x_2. Illustrate the progress of the algorithm graphically in x_1–x_2 space.