# Overview

Decomposition based approach.

Start with

- Easy constraints
- Complicating Constraints.

Put the complicating constraints into the objective and delete them from the constraints.

We will obtain a lower bound on the optimal solution for minimization problems.

In many situations, this bound is close to the optimal solution value.

# An Example: Constrained Shortest Paths

Given: a network G = (N,A)

$c_{ij}$ cost for arc (i,j)

$t_{ij}$ traversal time for arc (i,j)

z* = Min $$\sum_{(i,j)\in A} c_{ij} x_{ij}$$

s. t. $$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if i = 1} \\ -1 & \text{if i = n} \\ 0 & \text{otherwise} \end{cases}$$

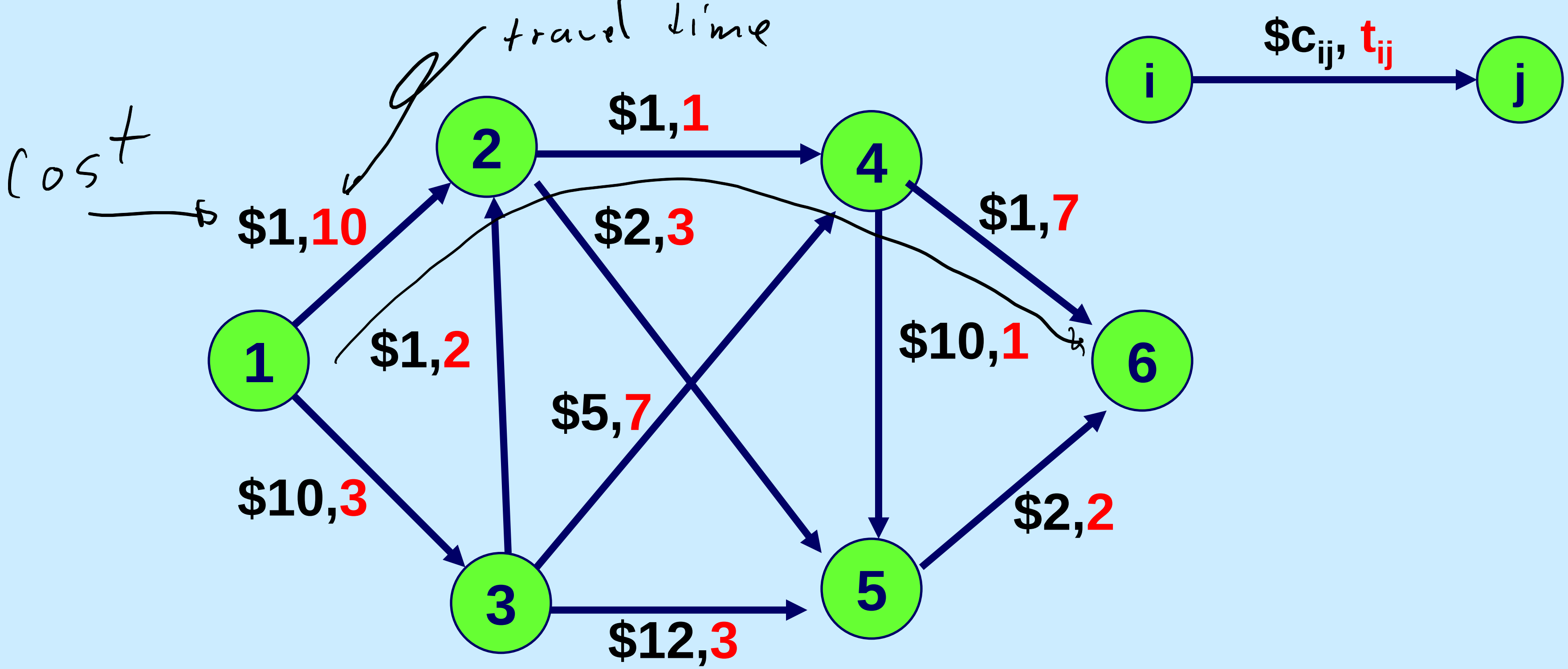
$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$$

Complicating constraint

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$$

# Example

**Find the shortest path from node 1 to node 6 with a transit time at most 10**



# Shortest Paths with Transit Time Restrictions

- **Shortest path problems are easy.**
- **Shortest path problems with transit time restrictions are NP-hard.**

**We say that constrained optimization problem Y is a *relaxation* of problem X if Y is obtained from X by eliminating one or more constraints.**

**We will "relax" the complicating constraint, and then use a "heuristic" of penalizing too much transit time. We will then connect it to the theory of Lagrangian relaxations.**

# Shortest Paths with Transit Time Restrictions

**<u>Step 1.</u> (A Lagrangian relaxation approach). Penalize violation of the constraint in the objective function.**

**z(λ) = Min** $\sum_{(i,j)\in A} c_{ij} x_{ij} + \lambda \left( \sum_{(i,j)\in A} t_{ij} x_{ij} - T \right)$

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$$

**Complicating constraint**

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$$

**Note: $z^*(\lambda) \le z^* \quad \forall \lambda \ge 0$**

# Shortest Paths with Transit Time Restrictions

**Step 2.** Delete the complicating constraint(s) from the problem. The resulting problem is called the ***Lagrangian relaxation***.

**L(λ) = Min** $\sum_{(i,j)\in A}(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if i = 1} \\ -1 & \text{if i = n} \\ 0 & \text{otherwise} \end{cases}$$

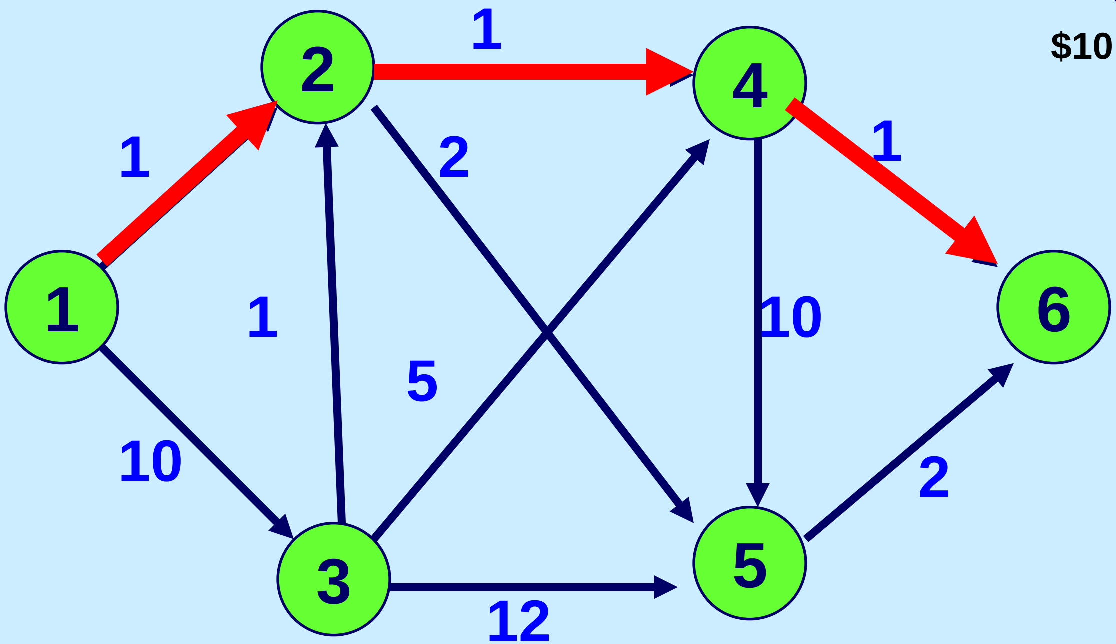
$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$$ **Complicating constraint**

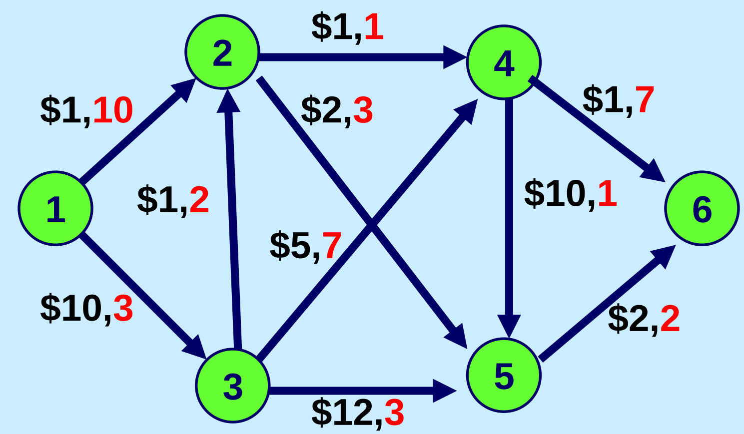
$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j)\in A$$

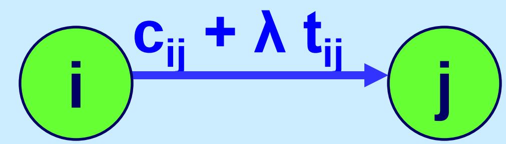
**Note: L(λ) ≤ z(λ) ≤ z*  ∀λ ≥ 0**

# What is the effect of varying λ?

Case 1: λ = 0




P =

c(P) =

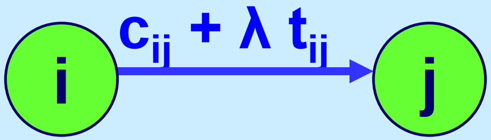
t(P) =

# Question to class

If $\lambda = 0$, the min cost path is found.

What happens to the (real) cost of the path as $\lambda$ increases from 0?

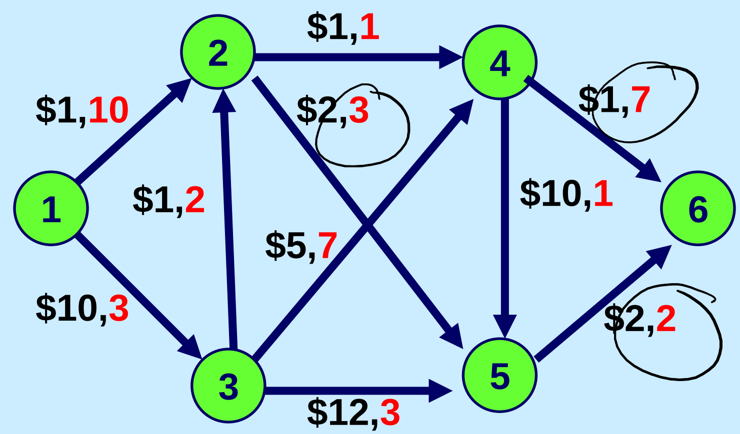
What path is determined as $\lambda$ gets VERY large?

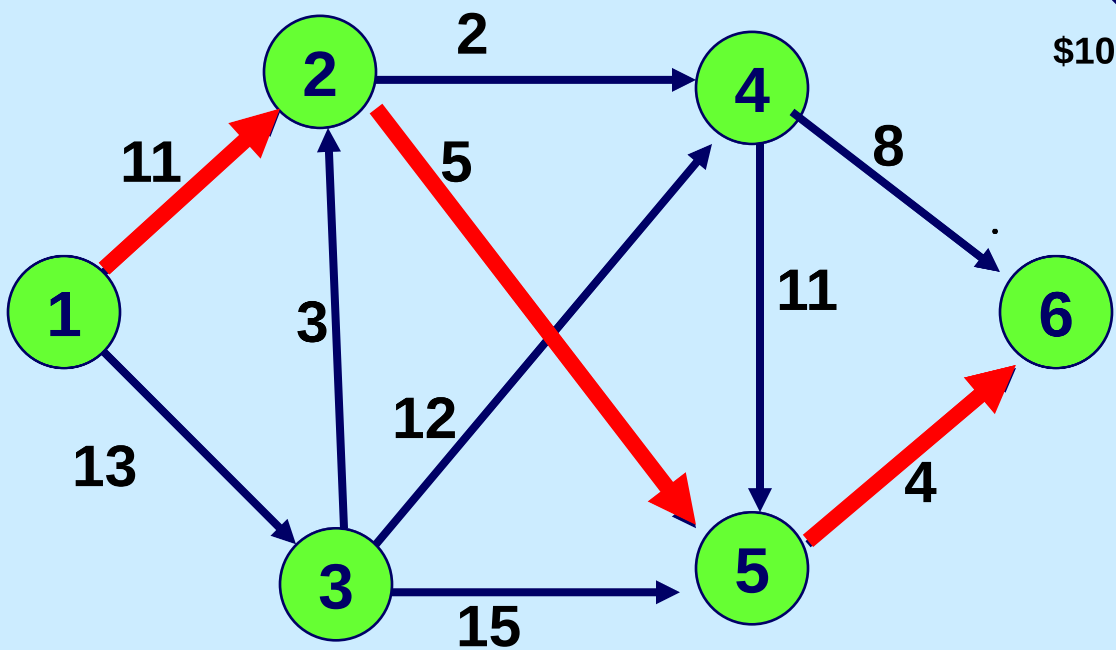



What happens to the (real) transit time of the path as $\lambda$ increases from 0?

# Let λ = 1

**Case 2: λ = 1**



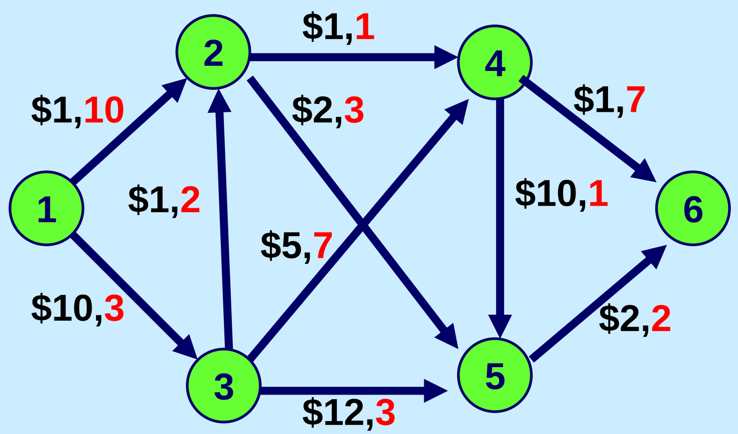

**P =**

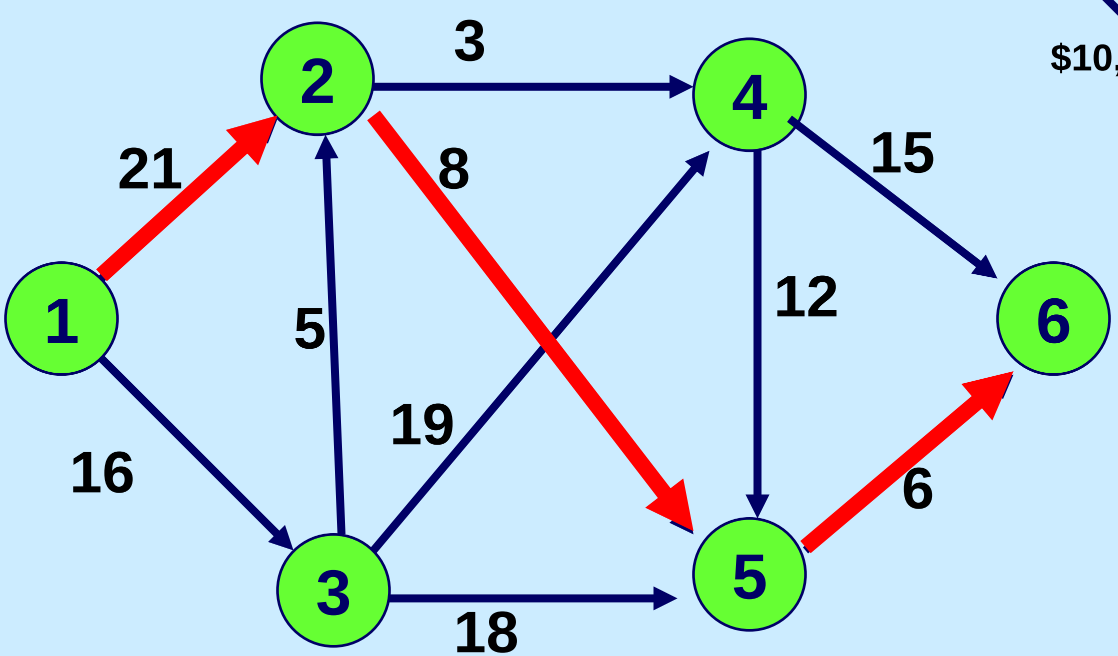
**c(P) =** 5

**t(P) =** 15

# Let λ = 2

**Case 3: λ = 2**



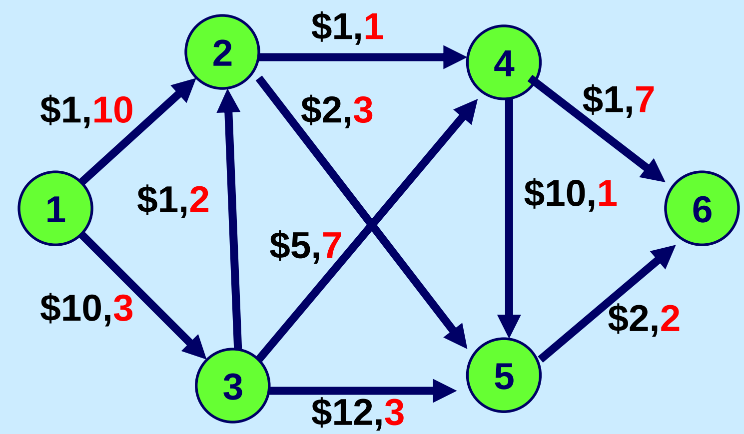



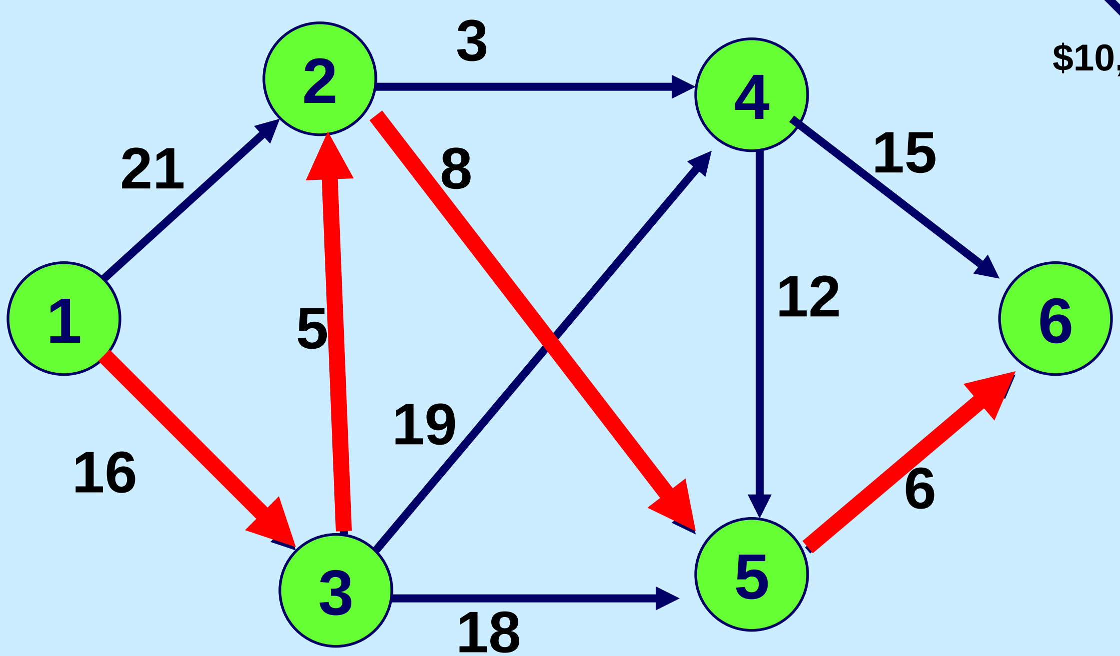

P =

c(P) = 5

t(P) = 15

# And alternative shortest path when λ = 2




P =
6

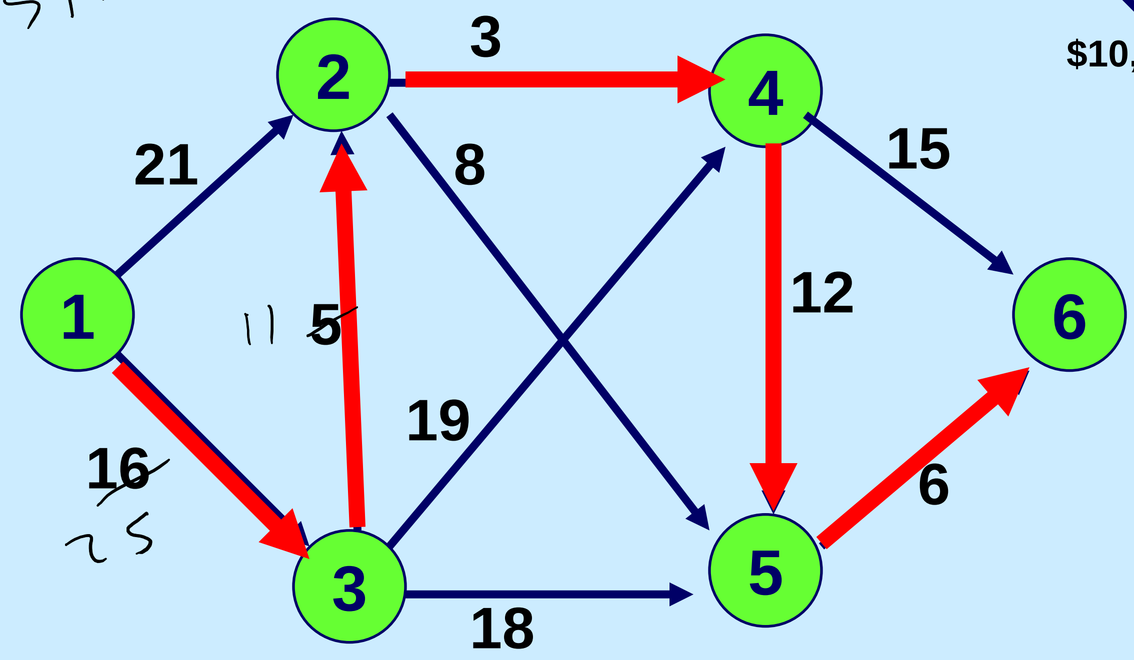
c(P) = 18

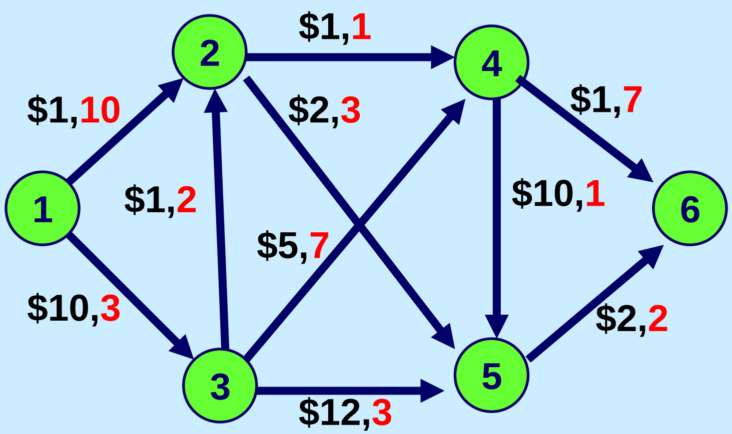
t(P) = 10

# Let λ = 5

**Case 4: λ = 5**

Error Still showing λ=2




| P = | c(P) = 24 | t(P) = 9 |
|---|---|---|

# A parametric analysis

| Toll | modified cost | Cost | Transit Time | Modified cost -10λ<br>A lower bound on z* |
|---|---|---|---|---|
| $0 \le \lambda \le 2/3$ | $3 + 18\lambda$ | 3 | 18 | $3 + 8\lambda$ |
| $2/3 \le \lambda \le 2$ | $5 + 15\lambda$ | 5 | 15 | $5 + 3\lambda$ |
| $2 \le \lambda \le 4.5$ | $15 + 10\lambda$ | 15 | 10 | 15 |
| $4.5 \le \lambda < \infty$ | $24 + 8\lambda$ | 24 | 8 | $24 - 2\lambda$ |

**The best value of λ is the one that maximizes the lower bound.**


Costs
Modified Cost – 10λ
Transit Times
25
20
15
10
5
0
0
5
10
λ


modified cost
120
100
80
60
40
20
0
0
2
4
6
8
10
λ

# The Lagrangian Multiplier Problem

L(λ) = min $\sum_{(i,j)\in A}(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$

s.t. $\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if i = 1} \\ -1 & \text{if i = n} \\ 0 & \text{otherwise} \end{cases}$

$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i, j) \in A$

L* = max {L(λ) : λ ≥ 0}. **Lagrangian Multiplier Problem**

**Theorem.** *L(λ) ≤ L* ≤ z*.*

# Application to constrained shortest path

L(λ) = min $\sum_{(i,j)\in A}(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$

Let c(P) be the cost of path P that satisfies the transit time constraint.

**Corollary.** For all λ, L(λ) ≤ L* ≤ z* ≤ c(P).

If L(λ') = c(P), then L(λ') = L* = z* = c(P). In this case, P is an optimal path and λ' optimizes the Lagrangian Multiplier Problem.

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