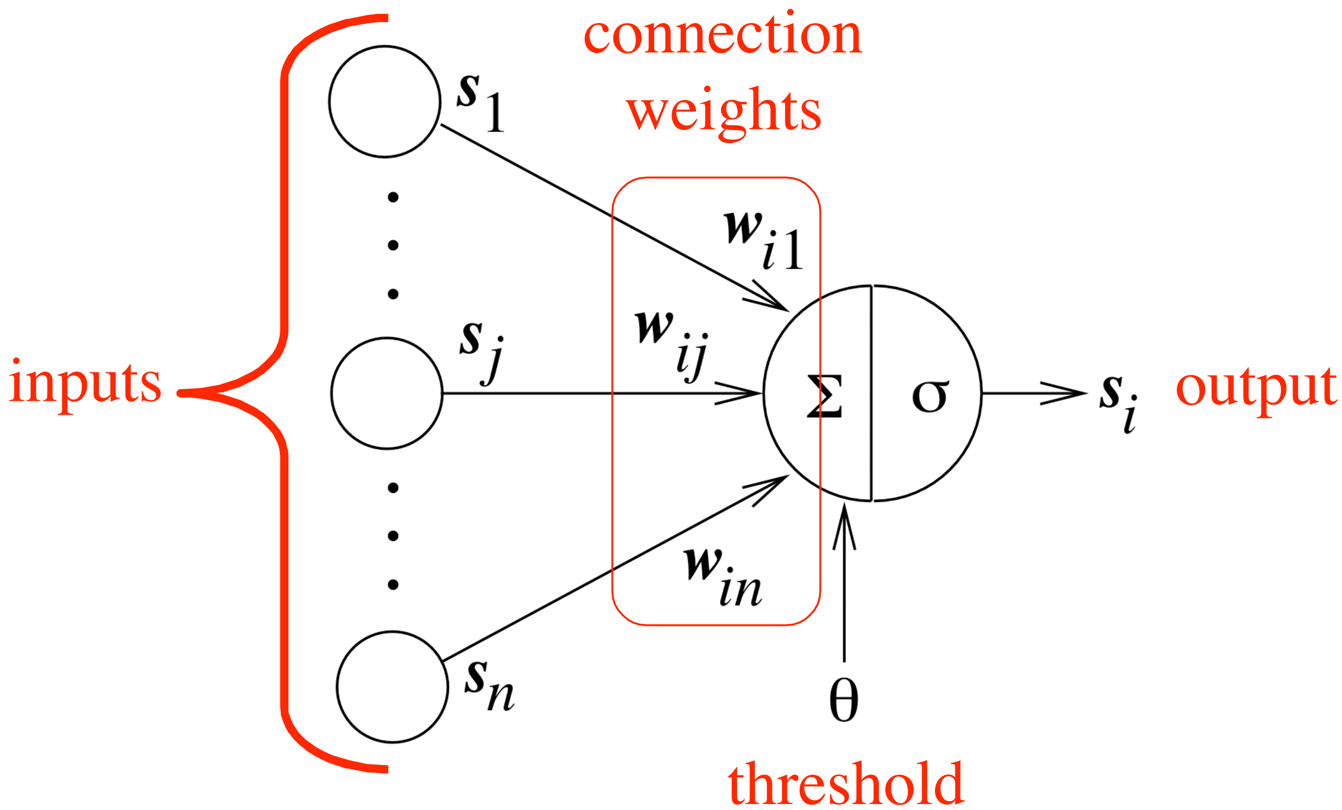
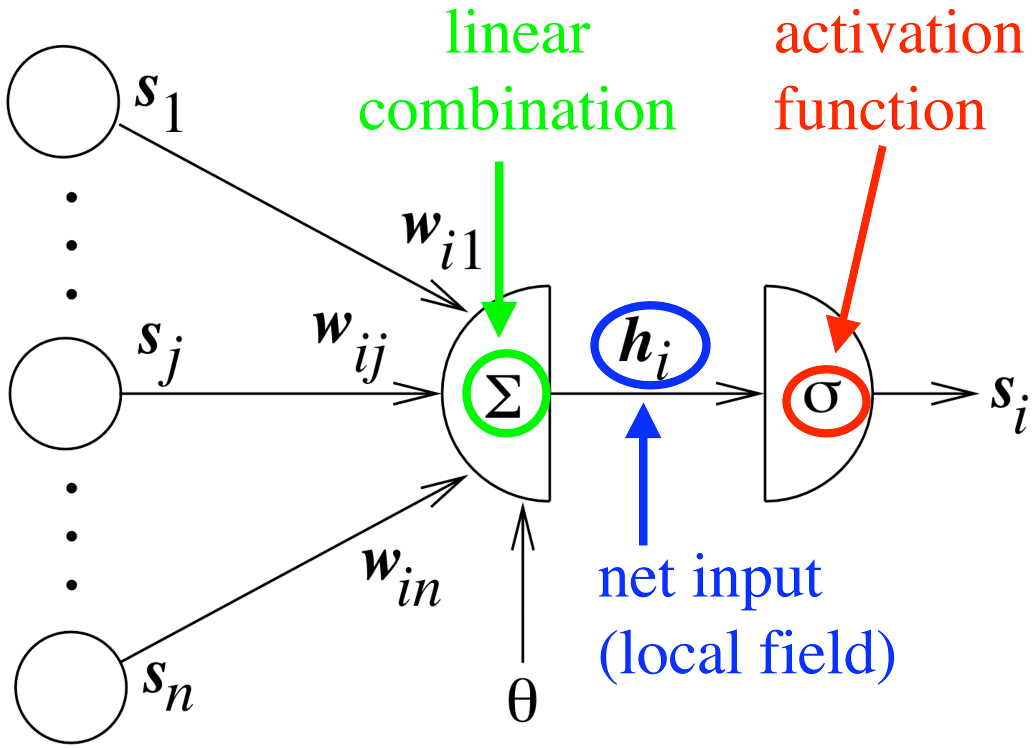
Artificial Neural Networks

(in particular, the Hopfield Network)

Typical Artificial Neuron



Typical Artificial Neuron



Equations

Net input:

$$h_i = \left(\sum_{j=1}^{n} w_{ij} s_j \right) - \theta$$

$$\mathbf{h} = \mathbf{W}\mathbf{s} - \theta$$

New neural state:

$$s_i' = \sigma(h_i)$$

$$\mathbf{s}' = \sigma(\mathbf{h})$$

Hopfield Network

- Symmetric weights: $w_{ij} = w_{ji}$
- No self-action: $w_{ii} = 0$
- Zero threshold: $\theta = 0$
- Bipolar states: $s_i \in \{-1, +1\}$
- Discontinuous bipolar activation function:

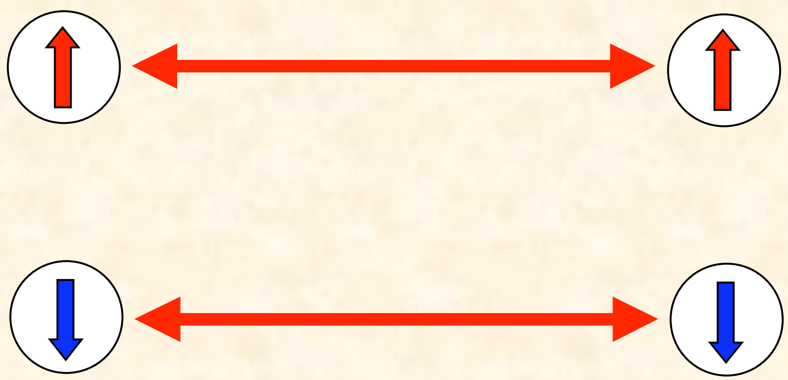
$$\sigma(h) = \operatorname{sgn}(h) = \begin{cases} -1, & h < 0 \\ +1, & h > 0 \end{cases}$$

What to do about $h = 0$?

- There are several options:
  - $\sigma(0) = +1$
  - $\sigma(0) = -1$
  - $\sigma(0) = -1$ or $+1$ with equal probability
  - $h_i = 0 \Rightarrow$ no state change ($s_i' = s_i$)
- Not much difference, but be consistent
- Last option is slightly preferable, since symmetric

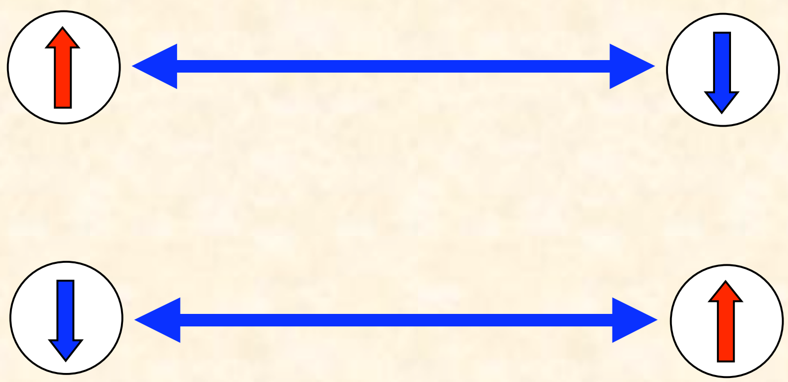
Positive Coupling

- Positive *sense* (sign)
- Large *strength*



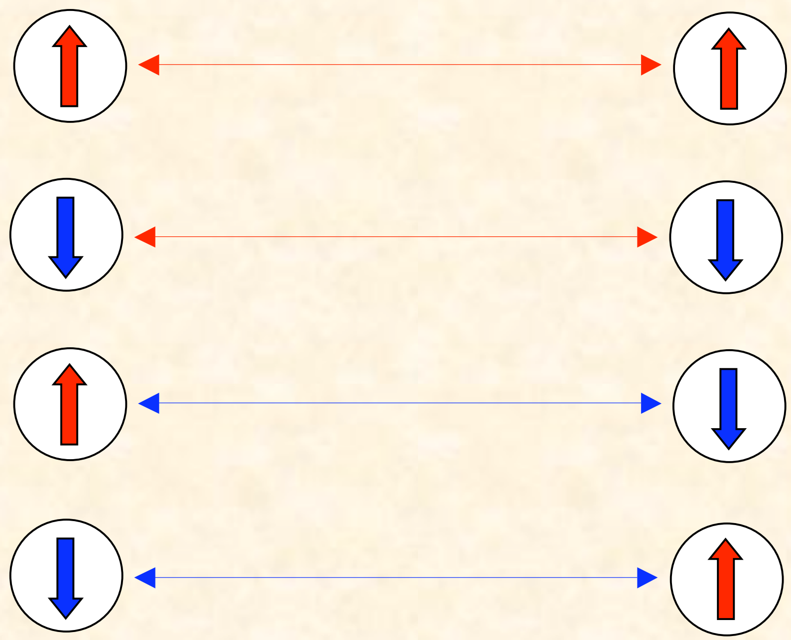
Negative Coupling

- Negative *sense* (sign)
- Large *strength*

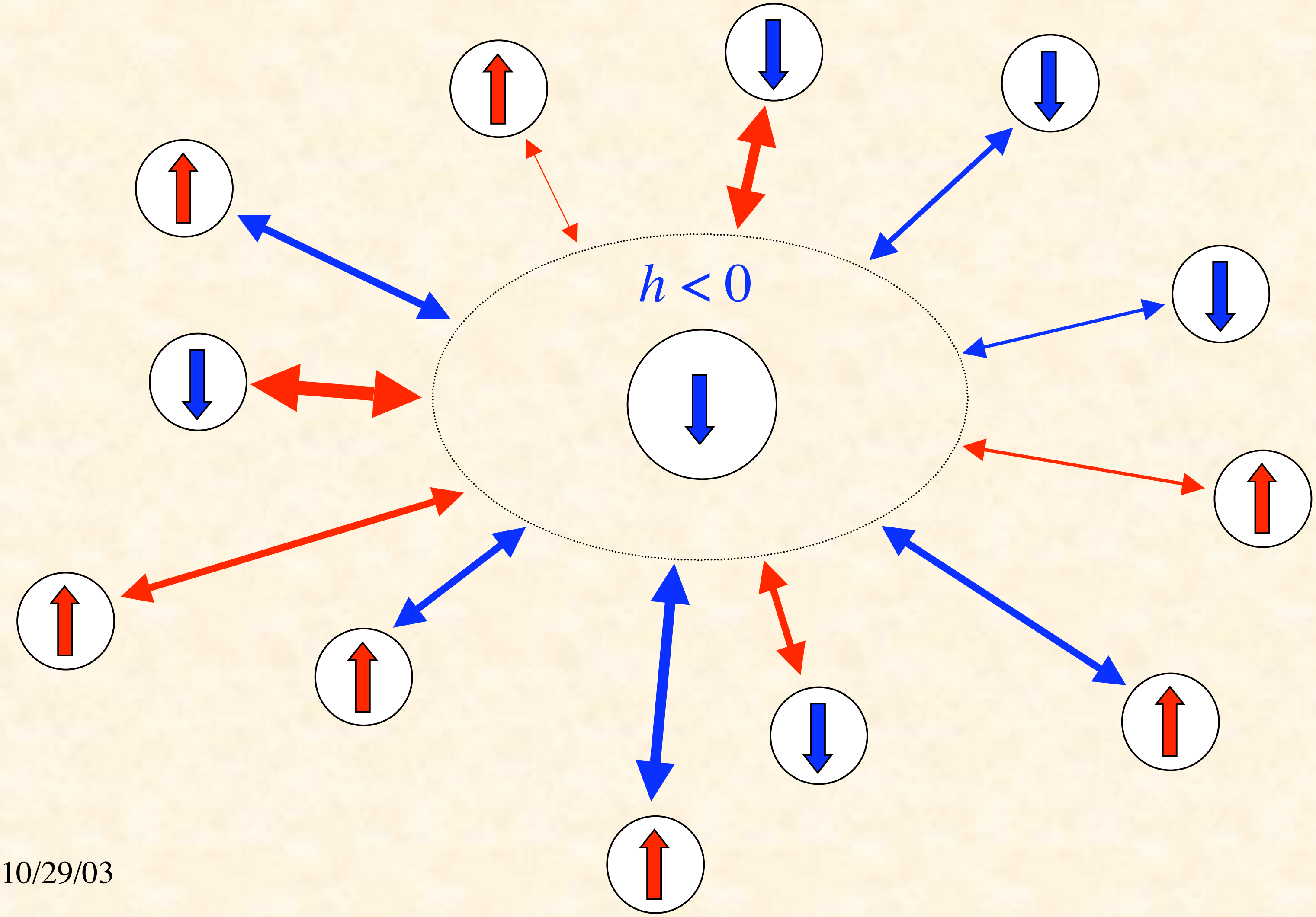


Weak Coupling

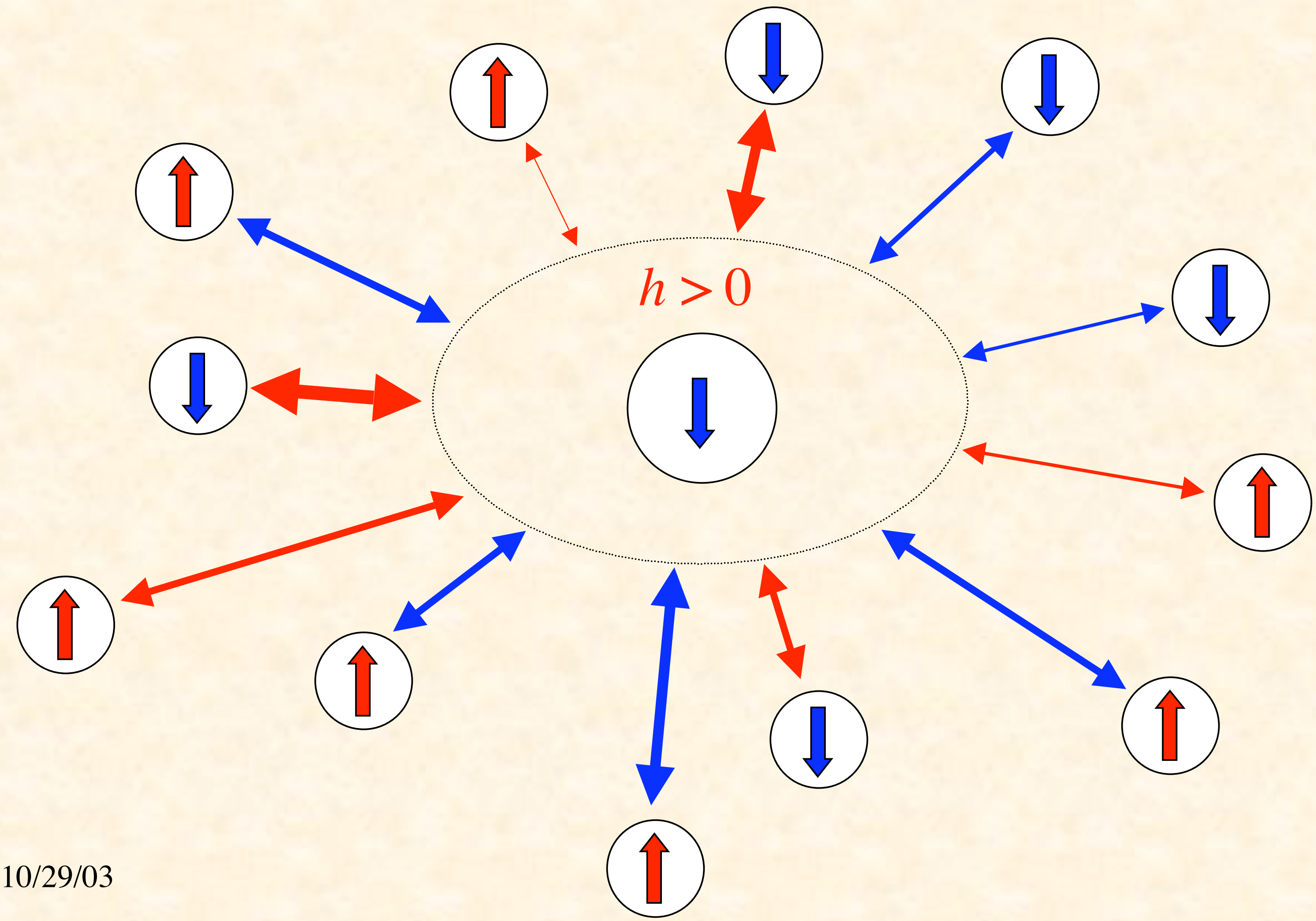
- Either *sense* (sign)
- Little *strength*



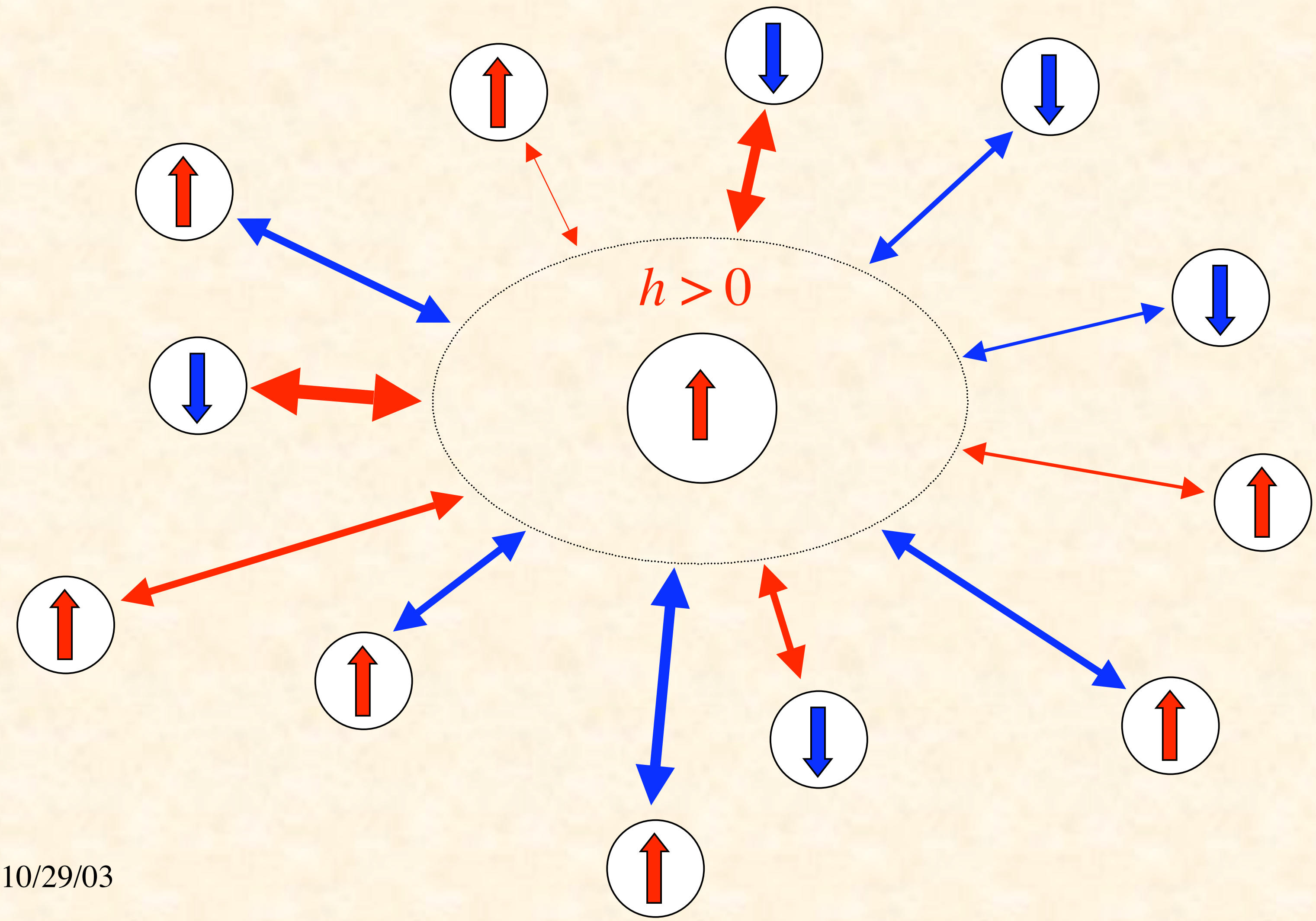
State = –1 & Local Field < 0



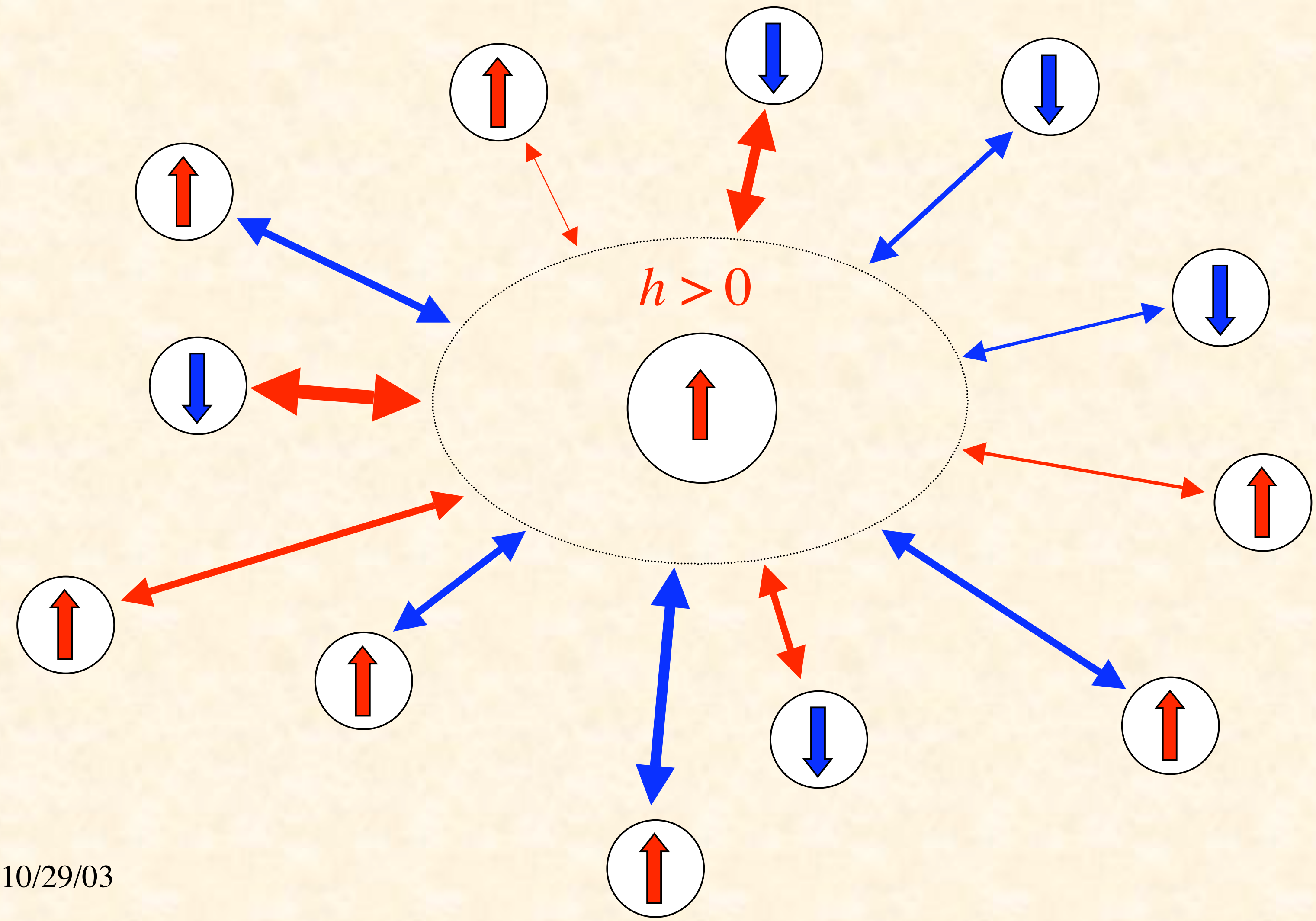
State = –1 & Local Field > 0



State Reverses



State = +1 & Local Field > 0



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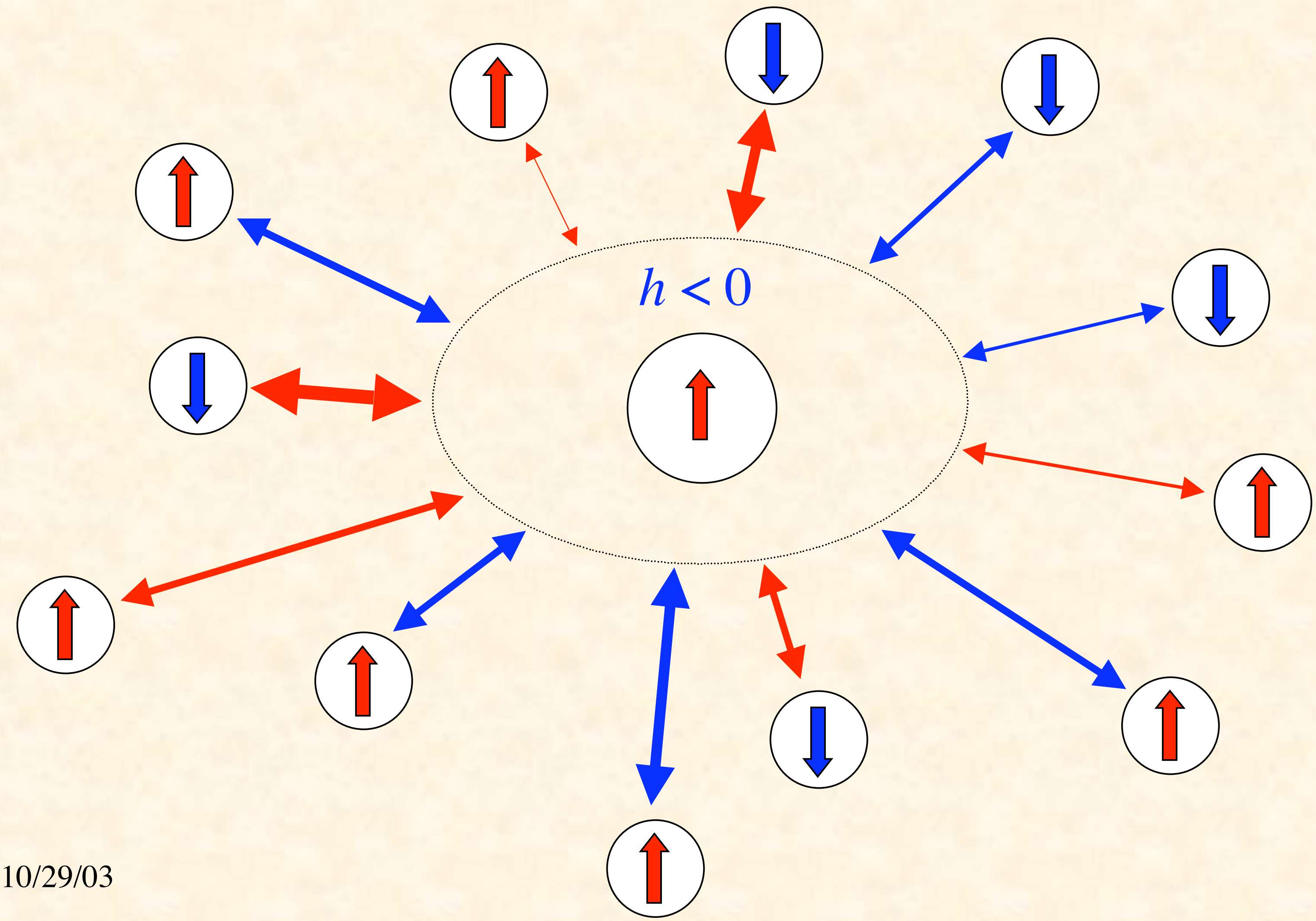
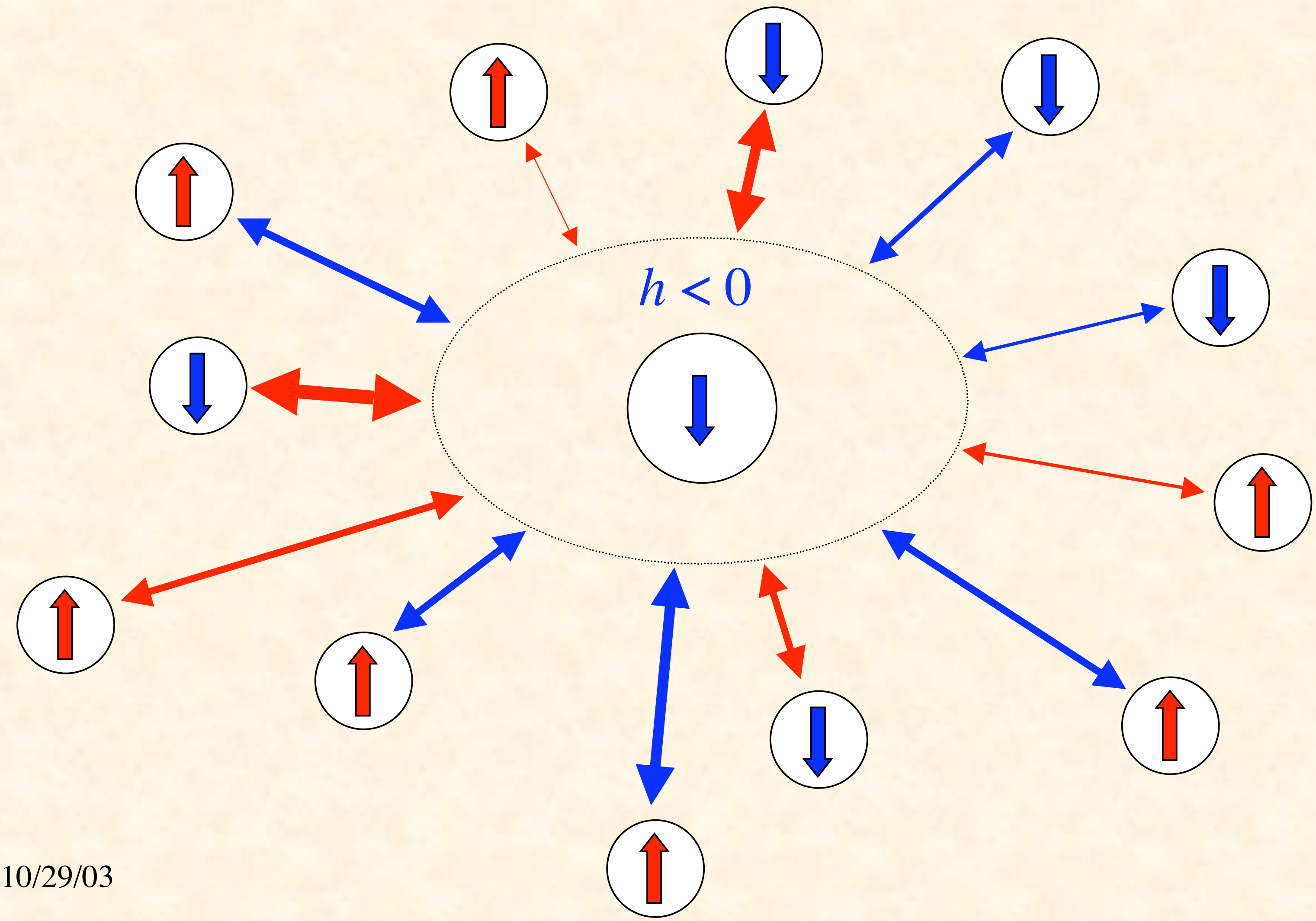


State Reverses



Hopfield Net as Soft Constraint Satisfaction System

- States of neurons as yes/no decisions
- Weights represent *soft constraints* between decisions
  - *hard* constraints *must* be respected
  - *soft* constraints have *degrees* of importance
- Decisions change to better respect constraints
- Is there an optimal set of decisions that best respects all constraints?

Convergence

- Does such a system converge to a stable state?
- Under what conditions does it converge?
- There is a sense in which each step relaxes the “tension” in the system
- But could a relaxation of one neuron lead to greater tension in other places?

Quantifying "Tension"

- If $w_{ij} > 0$, then s_i and s_j want to have the same sign ($s_i\, s_j = +1$)
- If $w_{ij} < 0$, then s_i and s_j want to have opposite signs ($s_i\, s_j = -1$)
- If $w_{ij} = 0$, their signs are independent
- Strength of interaction varies with $|w_{ij}|$
- Define disharmony ("tension") D_{ij} between neurons i and j:

$$D_{ij} = -\, s_i\, w_{ij}\, s_j$$

$D_{ij} < 0$ □ they are happy

$D_{ij} > 0$ □ they are unhappy

Total Energy of System

The "energy" of the system is the total "tension" (disharmony) in it:

$$E\{\mathbf{s}\} = \sum_{\langle ij \rangle} D_{ij} = -\sum_{\langle ij \rangle} s_i w_{ij} s_j$$

$$= -\tfrac{1}{2} \sum_{i} \sum_{j \neq i} s_i w_{ij} s_j$$

$$= -\tfrac{1}{2} \sum_{i} \sum_{j} s_i w_{ij} s_j$$

$$= -\tfrac{1}{2} \mathbf{s}^{\mathrm{T}} \mathbf{W} \mathbf{s}$$

Review of Some Vector Notation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^{\mathrm{T}}$$ (column vectors)

$$\mathbf{x}^{\mathrm{T}}\mathbf{y} = \sum_{i=1}^{n} x_i y_i = \mathbf{x} \cdot \mathbf{y}$$ (inner product)

$$\mathbf{x}\mathbf{y}^{\mathrm{T}} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix}$$ (outer product)

$$\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i M_{ij} y_j$$ (quadratic form)

Another View of Energy

The energy measures the number of neurons whose states are in disharmony with their local fields (i.e. of opposite sign):

$$
\begin{aligned}
E\{\mathbf{s}\} &= -\tfrac{1}{2}\sum_i \sum_j s_i w_{ij} s_j \\
&= -\tfrac{1}{2}\sum_i s_i \sum_j w_{ij} s_j \\
&= -\tfrac{1}{2}\sum_i s_i h_i \\
&= -\tfrac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{h}
\end{aligned}
$$

Do State Changes Decrease Energy?

- Suppose that neuron k changes state
- Change of energy:

$$\begin{aligned}
\Delta E &= E\{\mathbf{s}'\} - E\{\mathbf{s}\} \\
&= -\sum_{\langle ij \rangle} s'_i w_{ij} s'_j + \sum_{\langle ij \rangle} s_i w_{ij} s_j \\
&= -\sum_{j \neq k} s'_k w_{kj} s_j + \sum_{j \neq k} s_k w_{kj} s_j \\
&= -\left(s'_k - s_k\right) \sum_{j \neq k} w_{kj} s_j \\
&= -\Delta s_k h_k \\
&< 0
\end{aligned}$$

Energy Does Not Increase

- In each step in which a neuron is considered for update:
  $E\{\mathbf{s}(t+1)\} - E\{\mathbf{s}(t)\} \leq 0$
- Energy cannot increase
- Energy decreases if any neuron changes
- Must it stop?

Proof of Convergence in Finite Time

- There is a minimum possible energy:
  - The number of possible states $\mathbf{s} \in \{-1, +1\}^n$ is finite
  - Hence $E_{\min} = \min \left\{ E(\mathbf{s}) \mid \mathbf{s} \in \{\pm 1\}^n \right\}$ exists
- Must show it is reached in a finite number of steps

Steps are of a Certain Minimum Size

If $h_k > 0$, then (let h_{min} = min of possible positive h)

$$h_k \geq \min\left\{ h \,\middle|\, h = \sum_{j \neq k} w_{kj} s_j \wedge \mathbf{s} \in \{\pm\mathbf{1}\}^n \wedge h > 0 \right\} =_{df} h_{min}$$

$$\Delta E = -(\Delta s_k) h_k = -2h_k \leq -2h_{min}$$

If $h_k < 0$, then (let h_{max} = max of possible negative h)

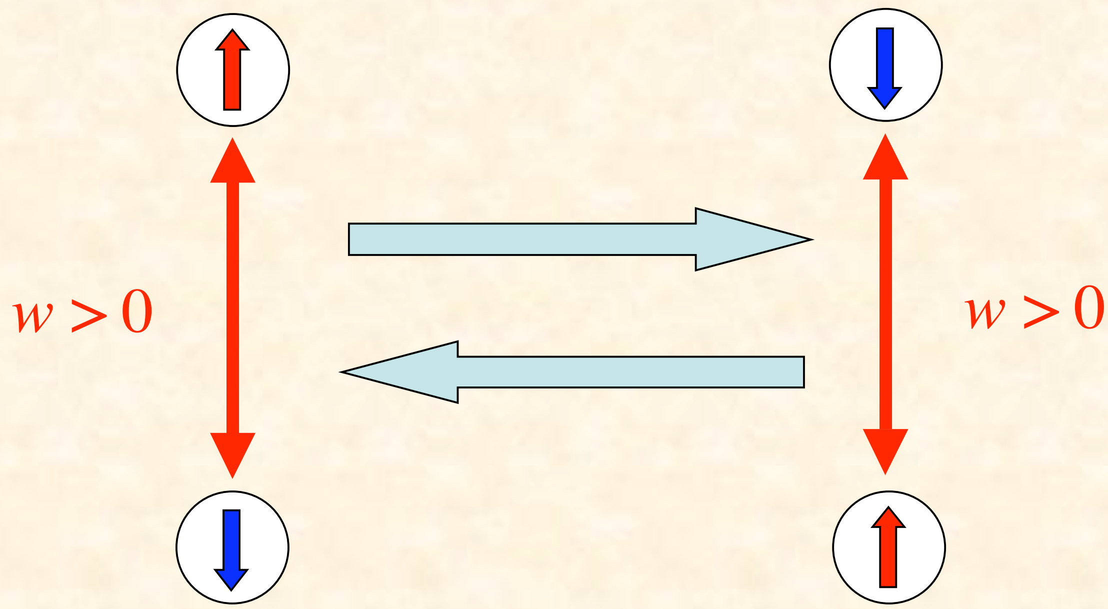
$$h_k \geq \max\left\{ h \,\middle|\, h = \sum_{j \neq k} w_{kj} s_j \wedge \mathbf{s} \in \{\pm\mathbf{1}\}^n \wedge h < 0 \right\} =_{df} h_{max}$$

$$\Delta E = -(\Delta s_k) h_k = 2h_k \leq 2h_{max}$$

Conclusion

- If we do asynchronous updating, the Hopfield net must reach a stable, minimum energy state in a finite number of updates
- This does not imply that it is a global minimum

Example Limit Cycle with Synchronous Updating

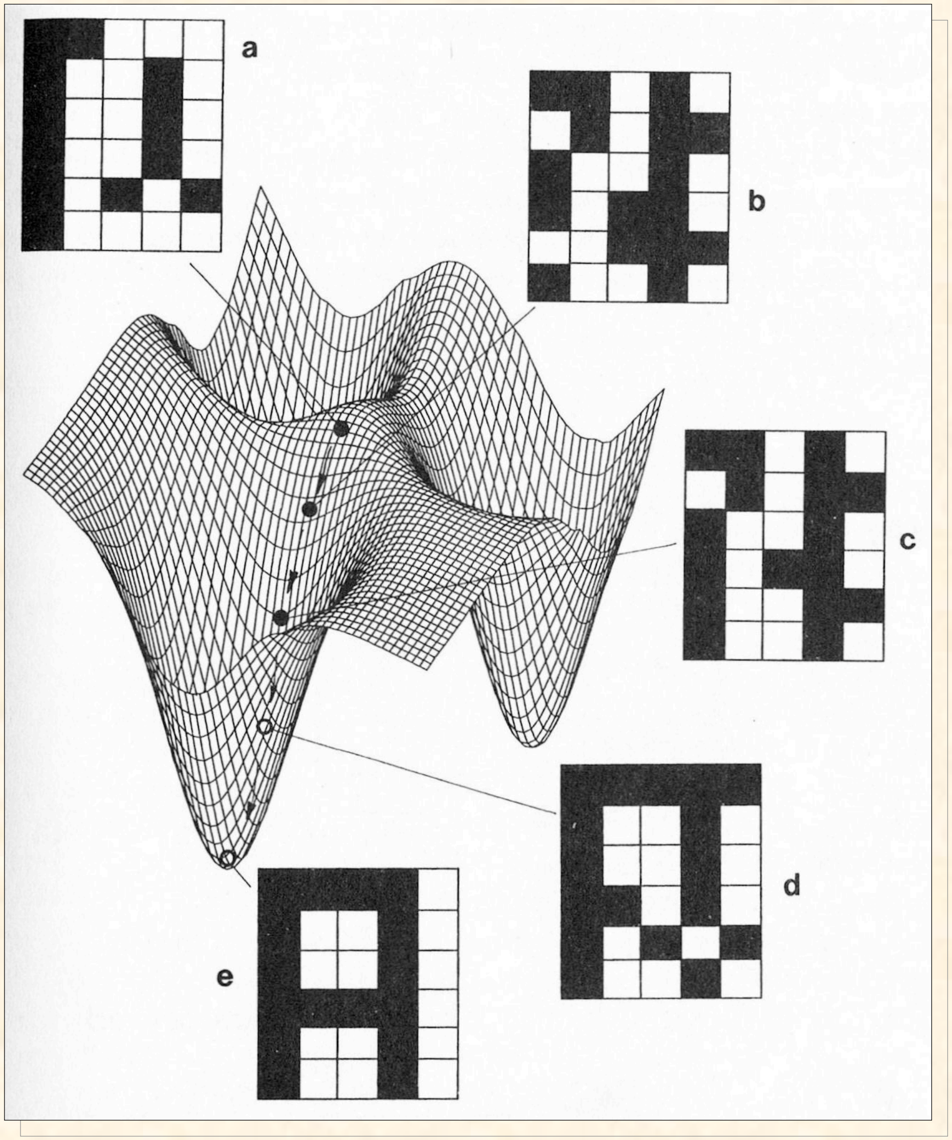


The Hopfield Energy Function is Even

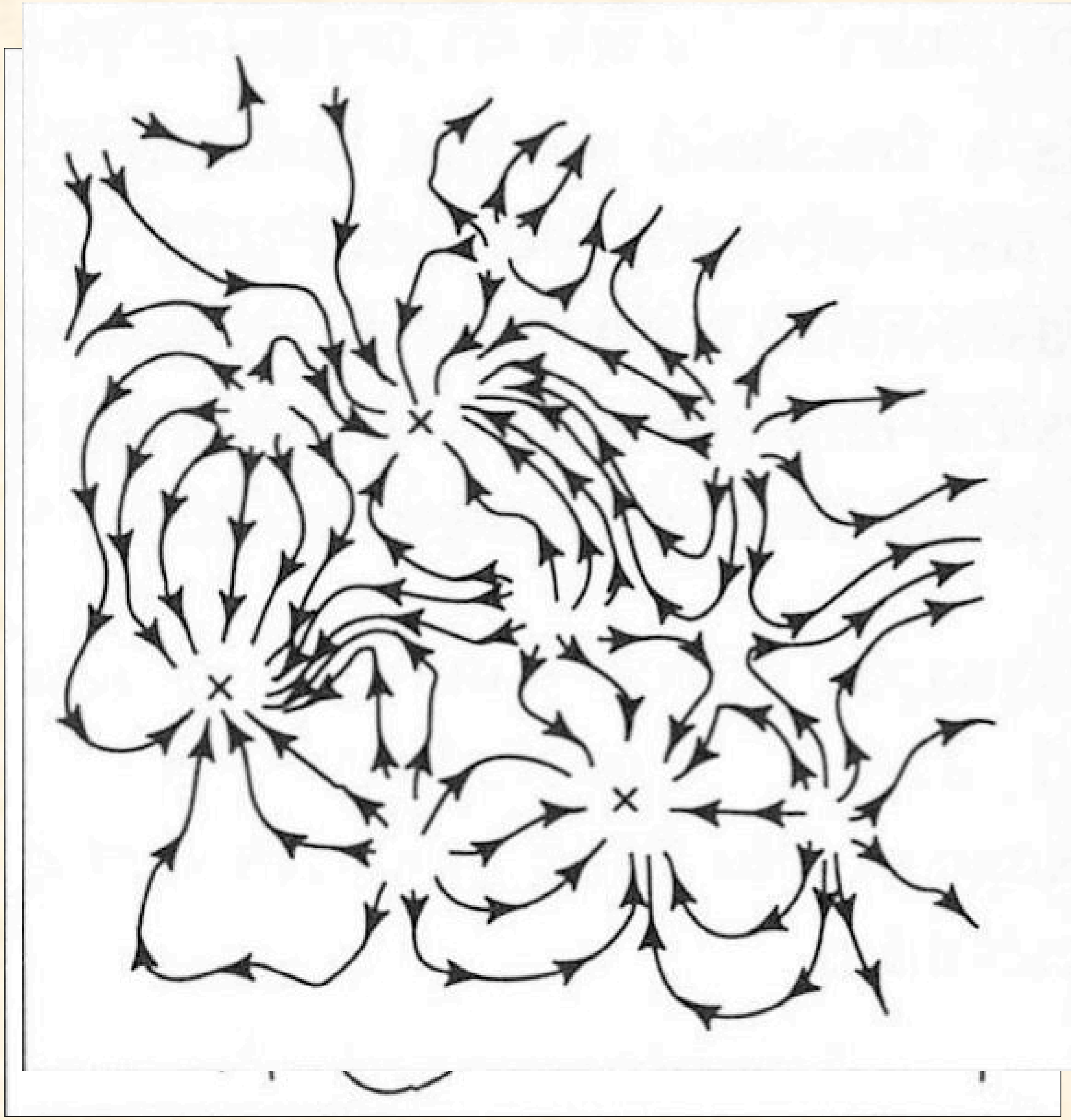
- A function f is **odd** if $f(-x) = -f(x)$, for all x
- A function f is **even** if $f(-x) = f(x)$, for all x
- Observe:

$$E\{-\mathbf{s}\} = -\tfrac{1}{2}(-\mathbf{s})^{\mathrm{T}}\mathbf{W}(-\mathbf{s}) = -\tfrac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{W}\mathbf{s} = E\{\mathbf{s}\}$$

Conceptual Picture of Descent on Energy Surface

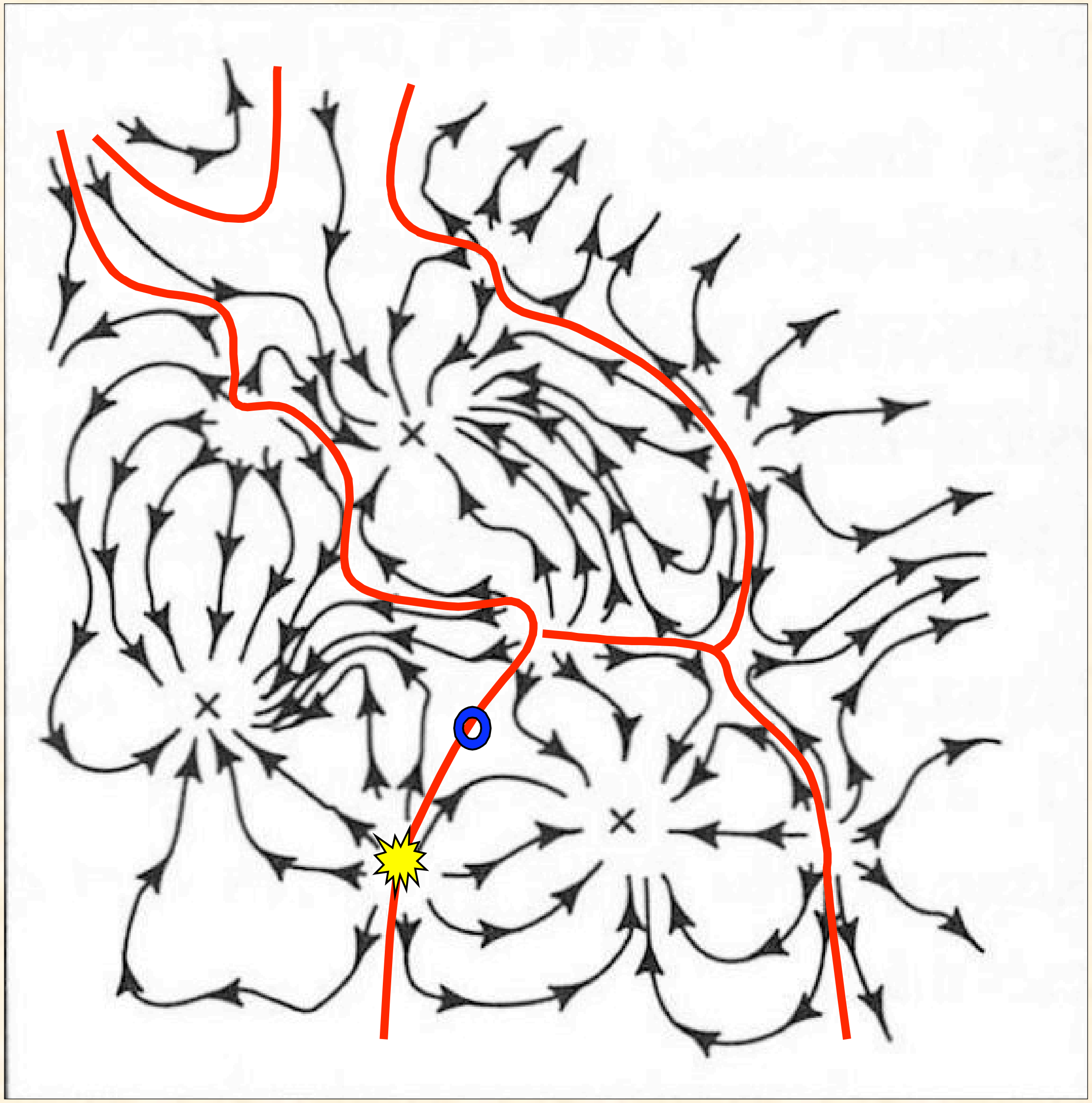


(fig. from Solé & Goodwin)



Energy Surface

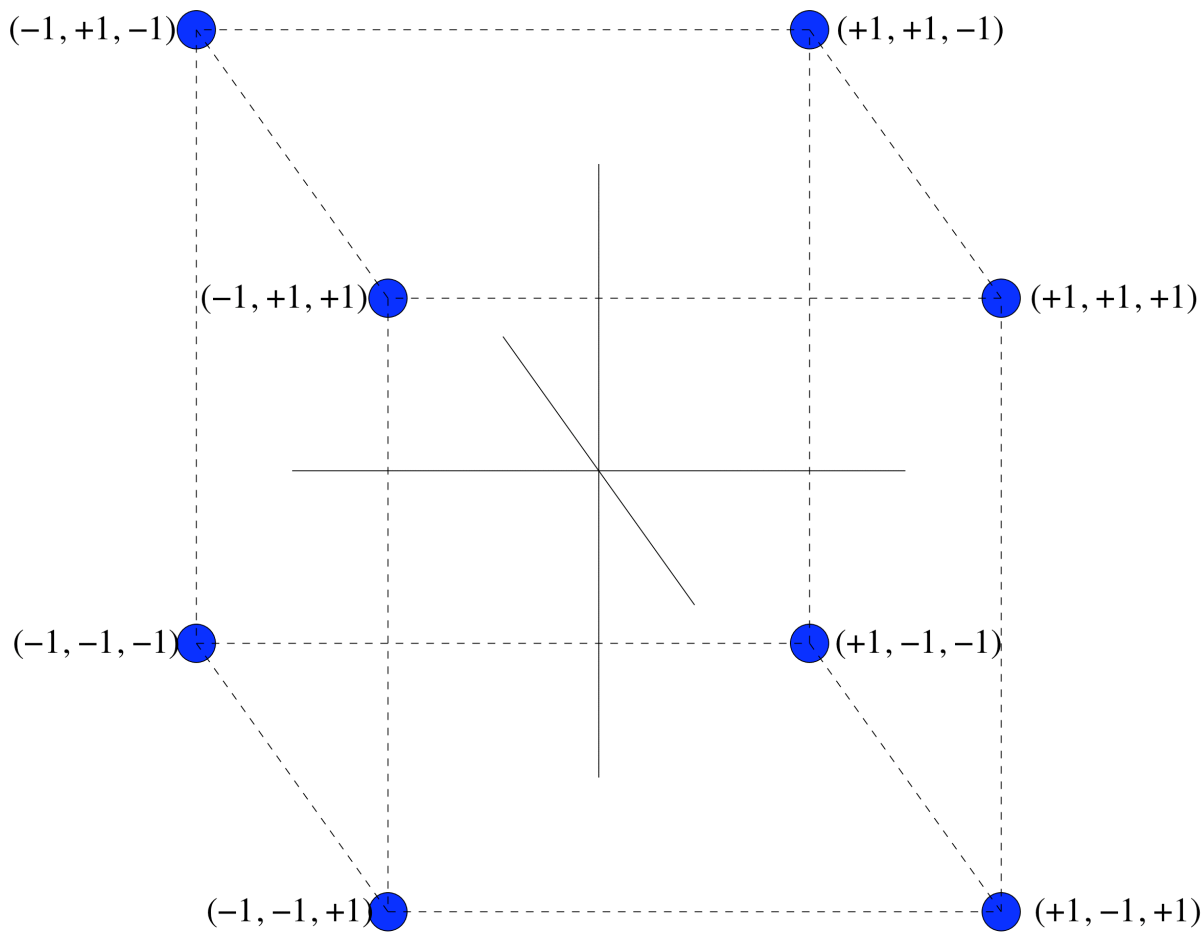
(fig. from Haykin *Neur*. *Netw*.)



Flow Lines

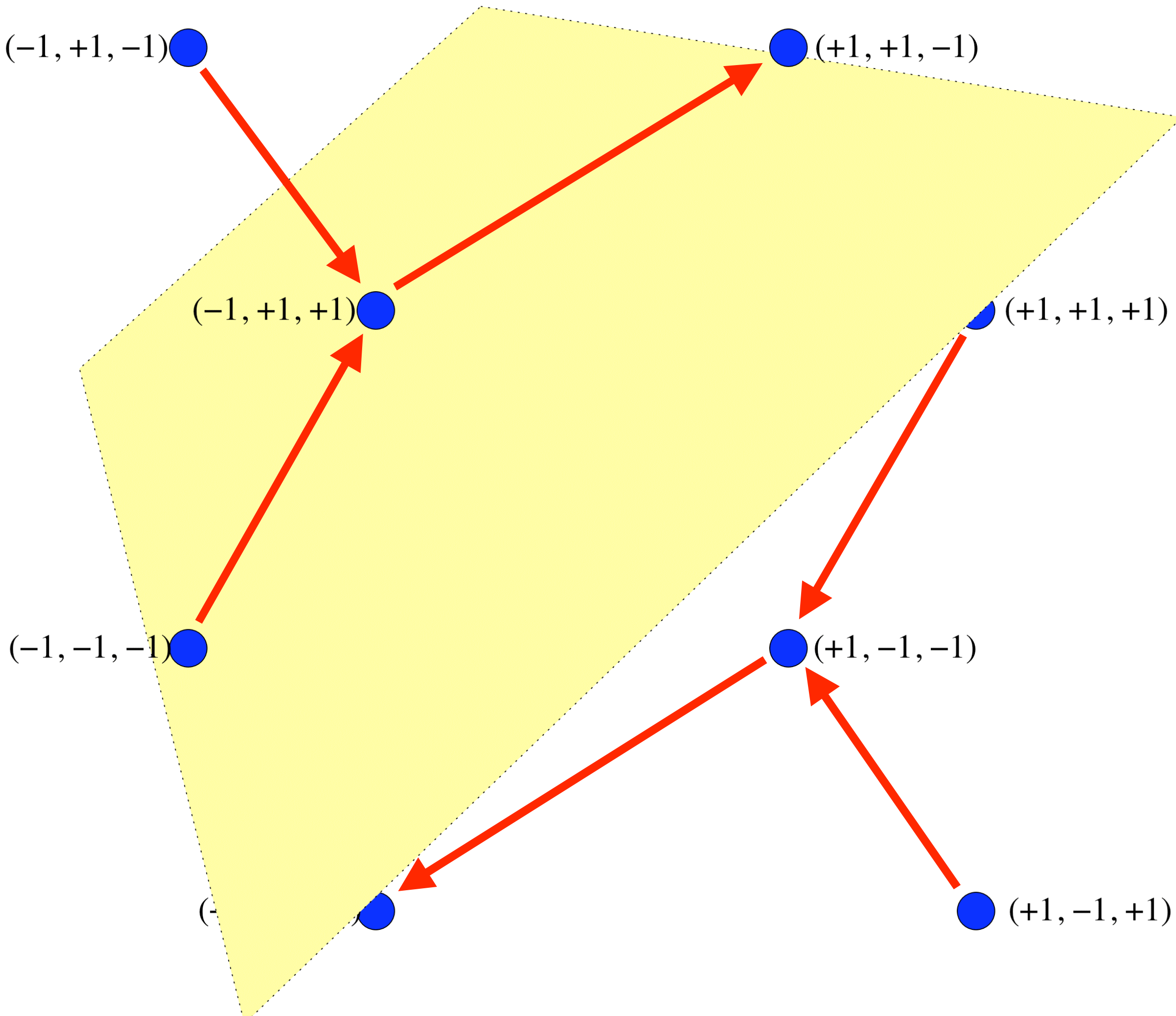
Basins of Attraction

(fig. from Haykin *Neur. Netw.*)



Bipolar State Space

Basins in Bipolar State Space



energy decreasing paths