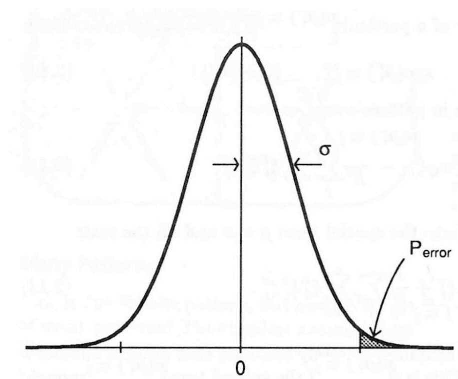
Capacity of Hopfield Memory

- Depends on the patterns imprinted
- If orthogonal, $p_{max} = n$
  - but every state is stable ⇒ trivial basins
- So $p_{max} < n$
- Let load parameter $\alpha = p / n$

equations

Single Bit Stability Analysis

- For simplicity, suppose $\mathbf{x}^k$ are random
- Then $\mathbf{x}^k \cdot \mathbf{x}^m$ are sums of n random ± 1
  - binomial distribution ≈ Gaussian
  - in range $-n, \ldots, +n$
  - with mean $\mu = 0$
  - and variance $\sigma^2 = n$
- Probability sum $> t$:



$$\tfrac{1}{2}\left[1 - \operatorname{erf}\left(\frac{t}{\sqrt{2n}}\right)\right]$$

[See “Review of Gaussian (Normal) Distributions” on course website]

Approximation of Probability

Let crosstalk $C_i^m = \frac{1}{n} \sum_{k \neq m} x_i^k \left(\mathbf{x}^k \cdot \mathbf{x}^m \right)$

We want $\Pr\left\{ C_i^m > 1 \right\} = \Pr\left\{ nC_i^m > n \right\}$

Note: $nC_i^m = \sum_{\substack{k=1 \\ k \neq m}}^{p} \sum_{j=1}^{n} x_i^k x_j^k x_j^m$

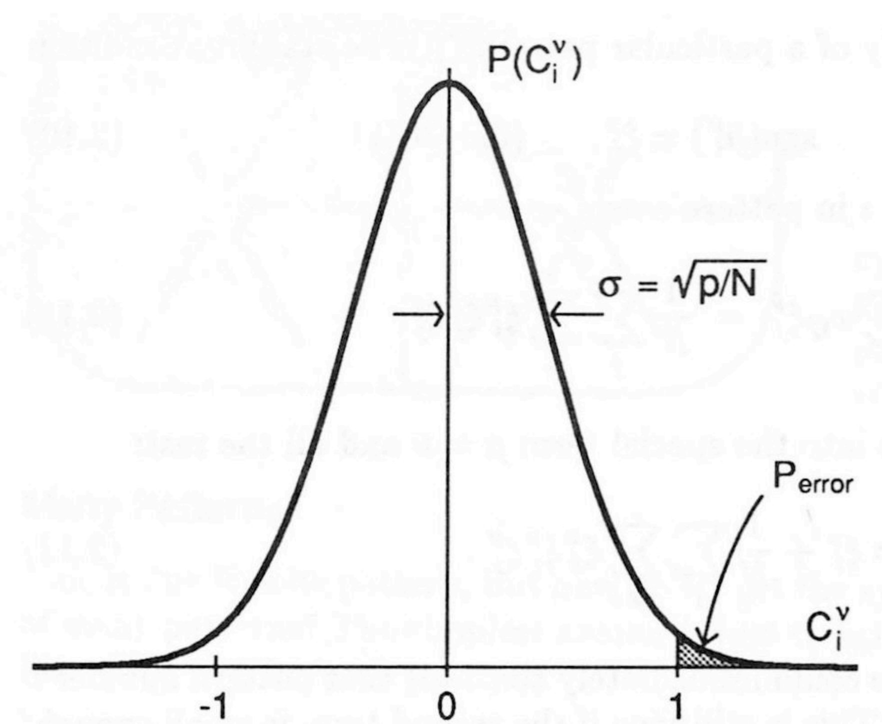
A sum of $n(p-1) \approx np$ random ± 1

Variance $\sigma^2 = np$

Probability of Bit Instability

$$\Pr\left\{nC_i^m > n\right\} = \tfrac{1}{2}\left[1 - \text{erf}\left(\frac{n}{\sqrt{2np}}\right)\right]$$

$$= \tfrac{1}{2}\left[1 - \text{erf}\left(\sqrt{n/2p}\right)\right]$$



 (fig. from Hertz & al. *Intr. Theory Neur. Comp.*)

Tabulated Probability of Single-Bit Instability

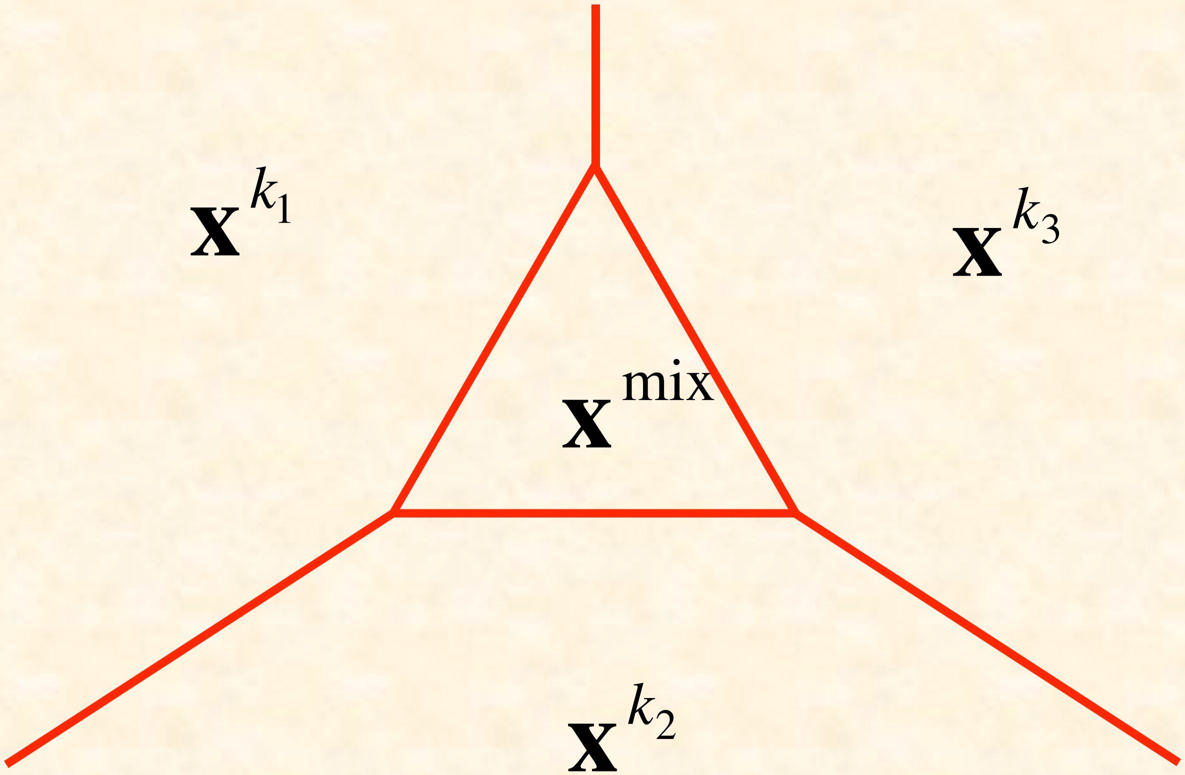
P_{error}	□
0.1%	0.105
0.36%	0.138
1%	0.185
5%	0.37
10%	0.61

(table from Hertz & al. *Intr. Theory Neur. Comp.*)

Spurious Attractors

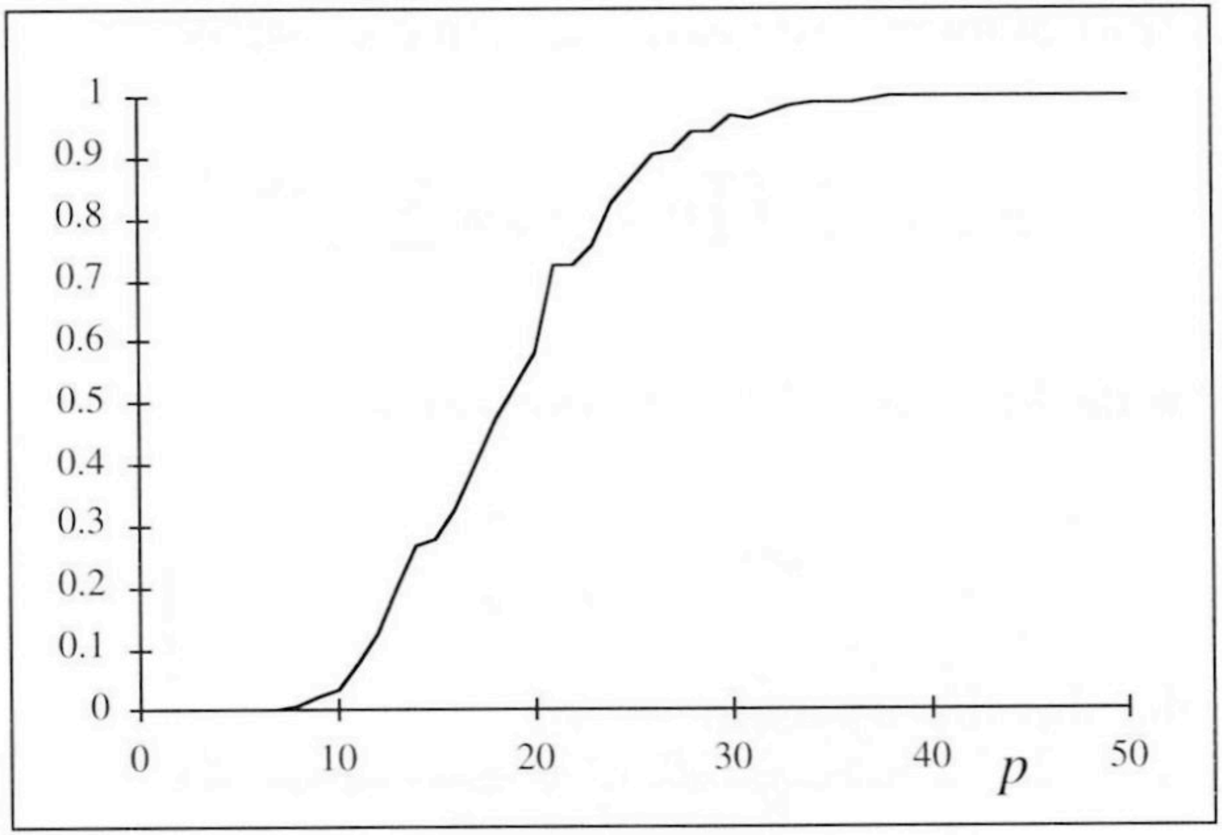
- Mixture states:
  - sums or differences of odd numbers of retrieval states
  - number increases combinatorially with *p*
  - shallower, smaller basins
  - basins of mixtures swamp basins of retrieval states □ overload
  - useful as combinatorial generalizations?
  - self-coupling generates spurious attractors
- Spin-glass states:
  - not correlated with any finite number of imprinted patterns
  - occur beyond overload because weights effectively random

Basins of Mixture States



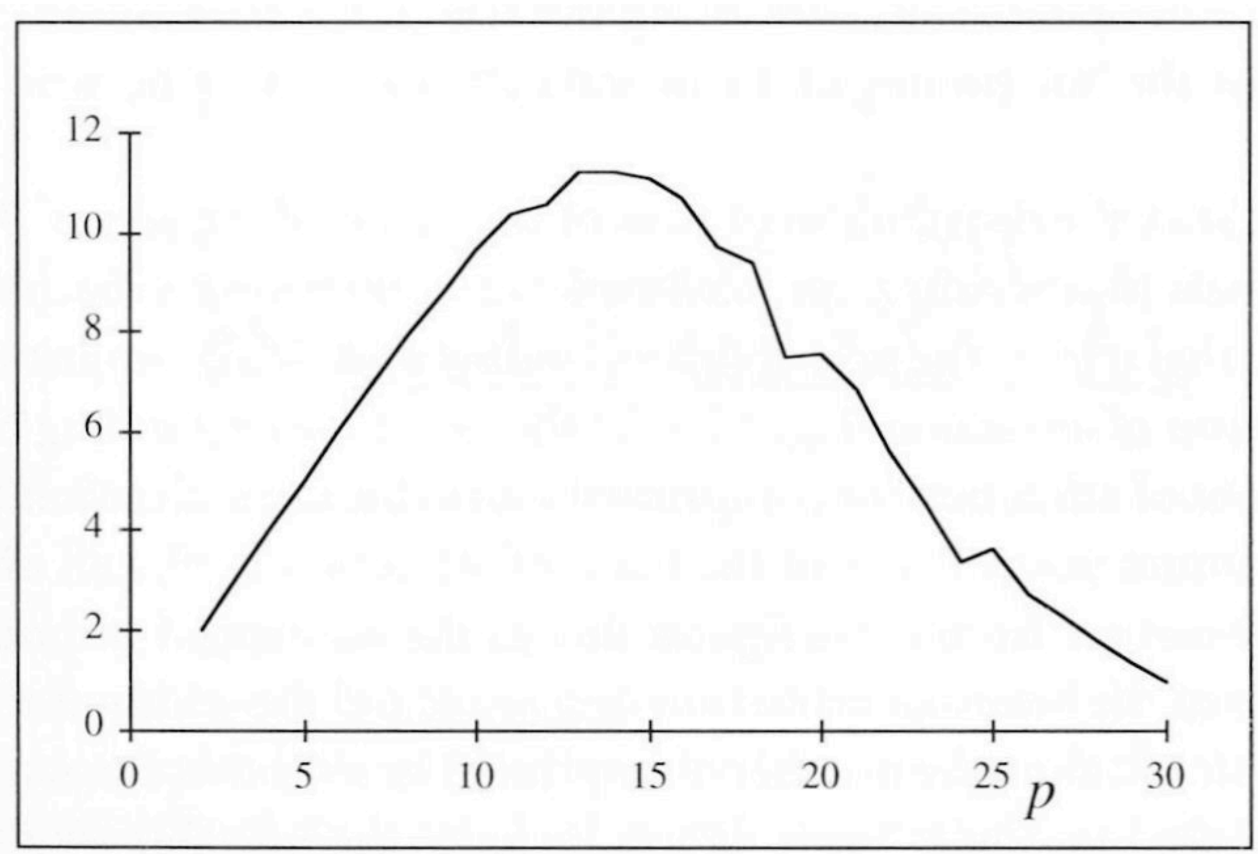
$$x_i^{\text{mix}} = \text{sgn}\left(x_i^{k_1} + x_i^{k_2} + x_i^{k_3}\right)$$

Fraction of Unstable Imprints ($n = 100$)

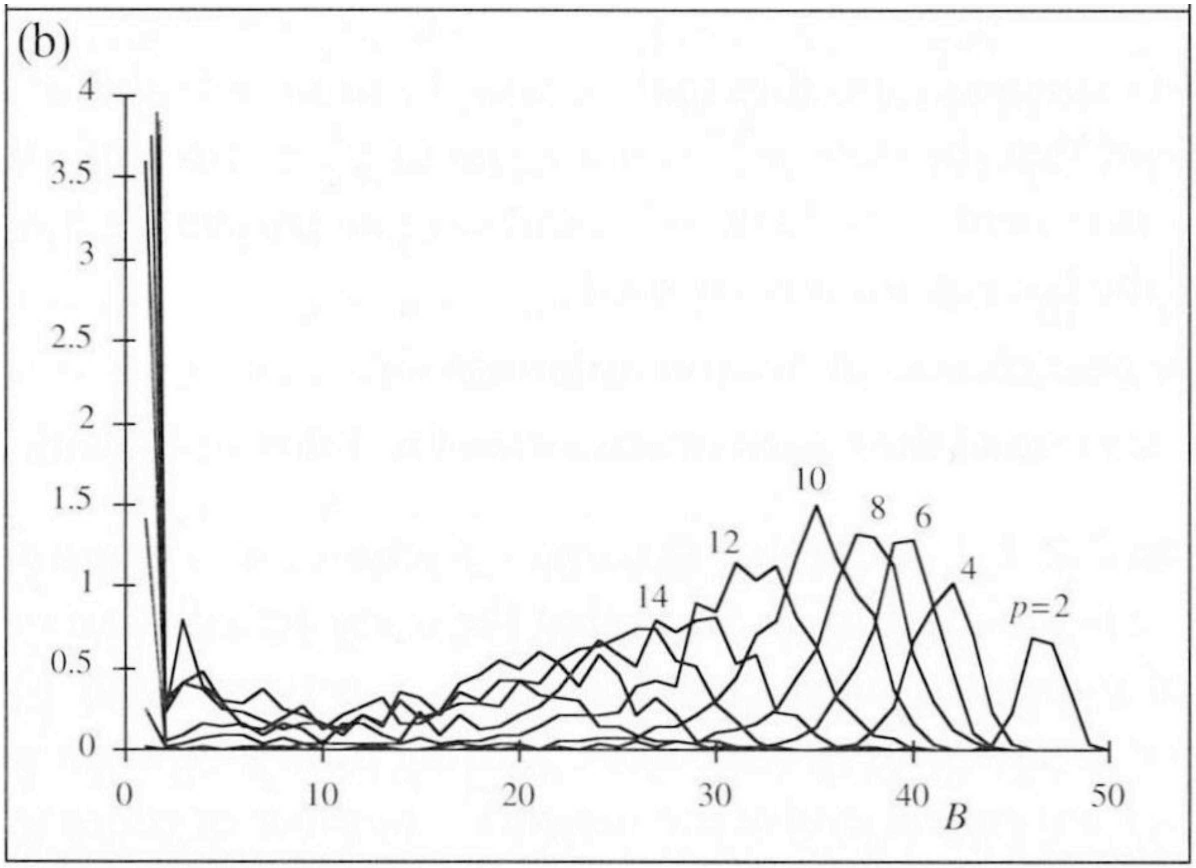


(fig from Bar-Yam)

Number of Stable Imprints (n = 100)



Number of Imprints with Basins of Indicated Size (n = 100)



(fig from Bar-Yam)

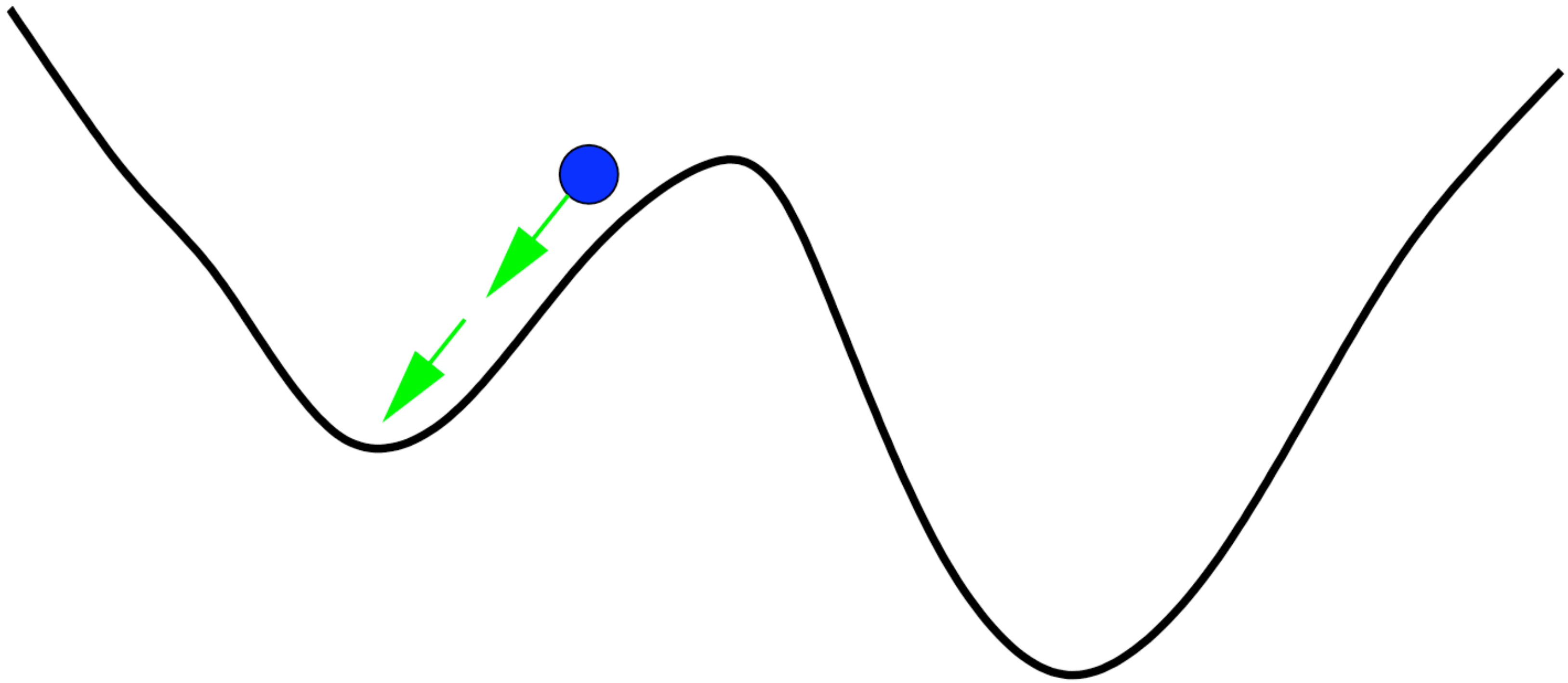
Summary of Capacity Results

- Absolute limit: $p_{max} < \alpha_c n = 0.138\ n$
- If a small number of errors in each pattern permitted: $p_{max} \propto n$
- If all or most patterns must be recalled perfectly: $p_{max} \propto n / \log n$
- Recall: all this analysis is based on *random* patterns
- Unrealistic, but sometimes can be arranged

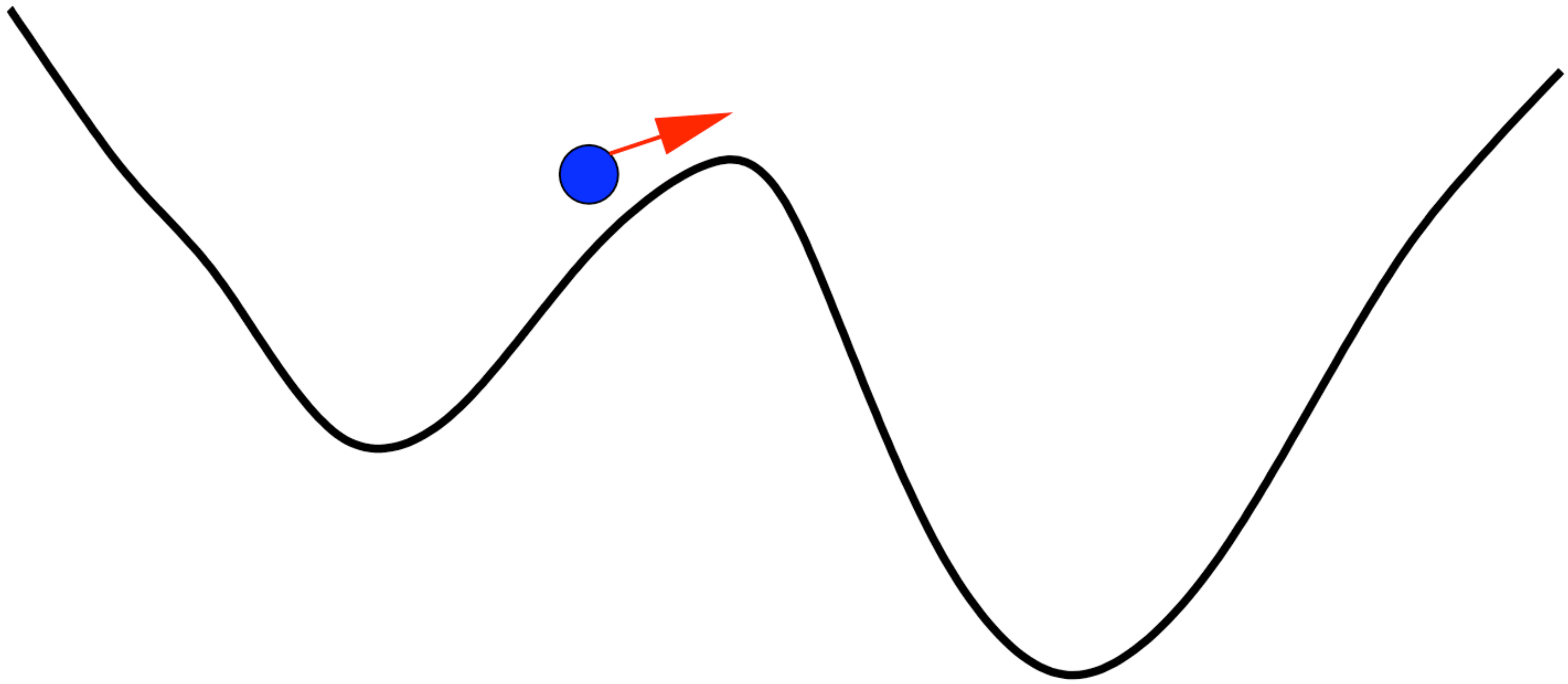
Stochastic Neural Networks

(in particular, the stochastic Hopfield network)

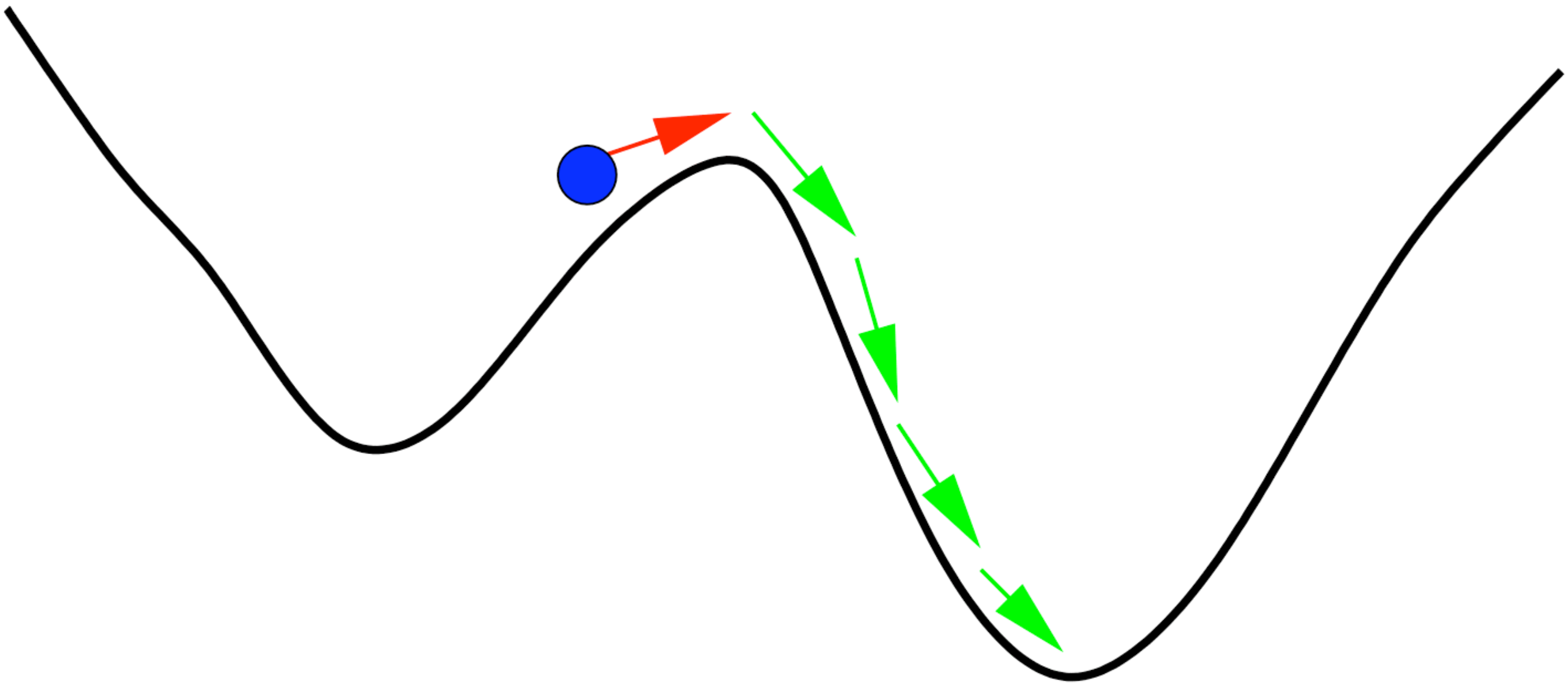
Trapping in Local Minimum



Escape from Local Minimum



Escape from Local Minimum



Motivation

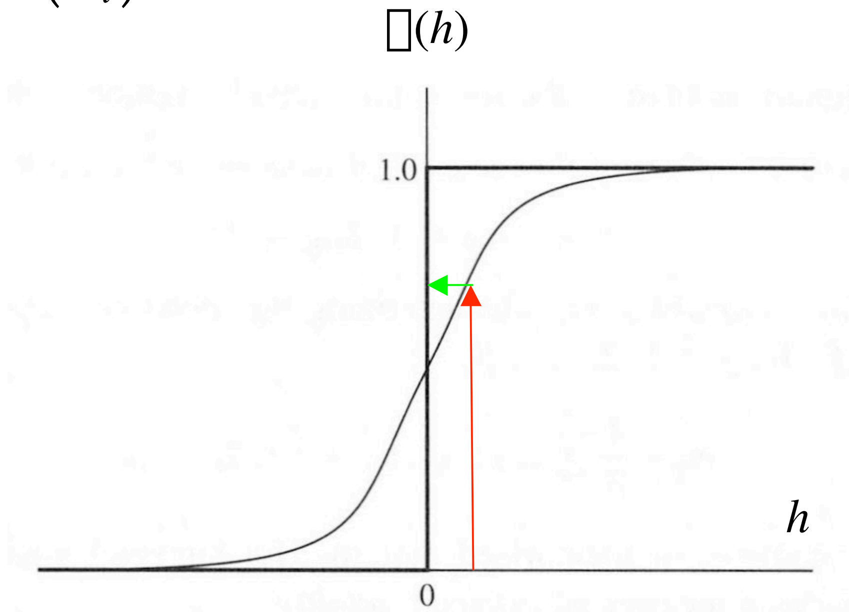
- Idea: with low probability, go against the local field
  - move up the energy surface
  - make the "wrong" microdecision
- Potential value for optimization: escape from local optima
- Potential value for associative memory: escape from spurious states
  - because they have higher energy than imprinted states

The Stochastic Neuron

Deterministic neuron: $s_i' = \operatorname{sgn}(h_i)$

$$\Pr\{s_i' = +1\} = \Theta(h_i)$$

$$\Pr\{s_i' = -1\} = 1 - \Theta(h_i)$$



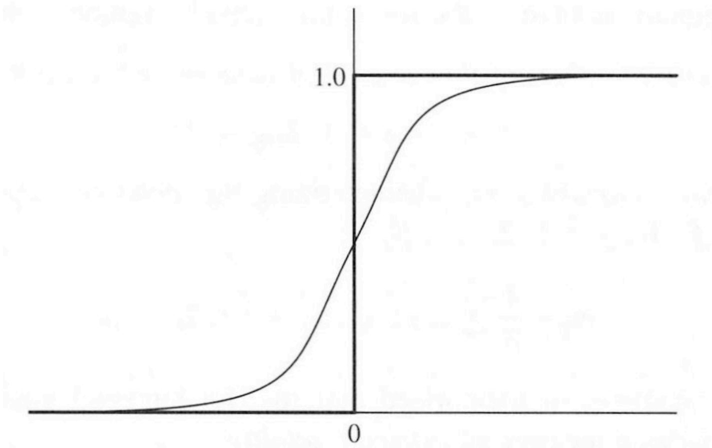
Stochastic neuron:

$$\Pr\{s_i' = +1\} = \sigma(h_i)$$

$$\Pr\{s_i' = -1\} = 1 - \sigma(h_i)$$

Logistic sigmoid: $\sigma(h) = \dfrac{1}{1 + \exp(-2h/T)}$

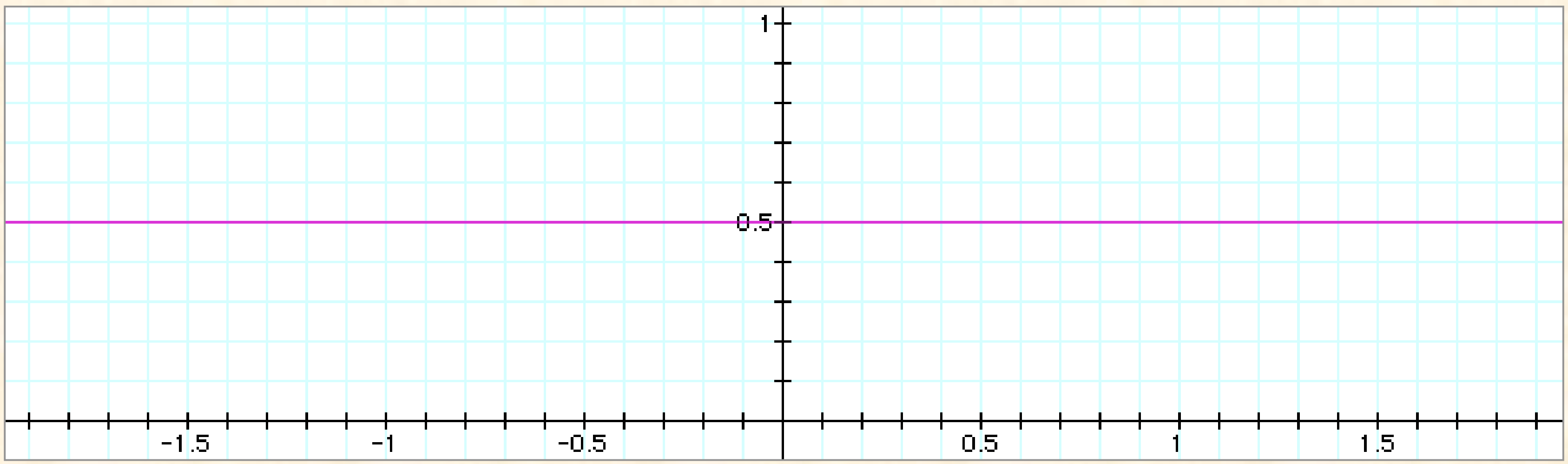
Properties of Logistic Sigmoid



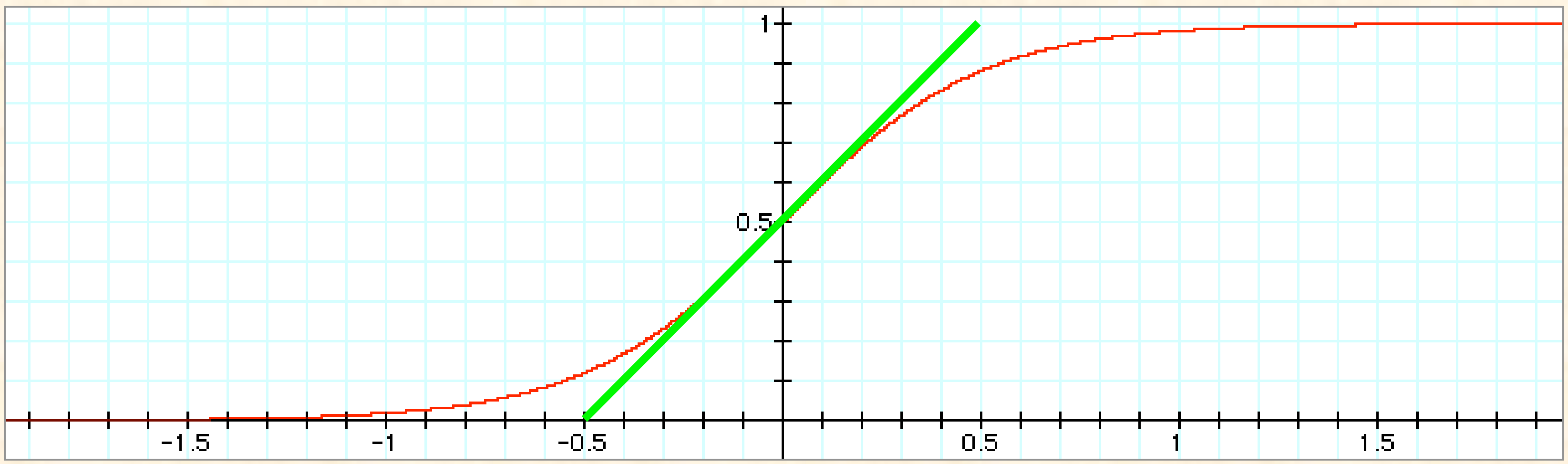
$$\sigma(h) = \frac{1}{1 + e^{-2h/T}}$$

- As $h \to +\infty$, $\sigma(h) \to 1$
- As $h \to -\infty$, $\sigma(h) \to 0$
- $\sigma(0) = 1/2$

Logistic Sigmoid With Varying *T*

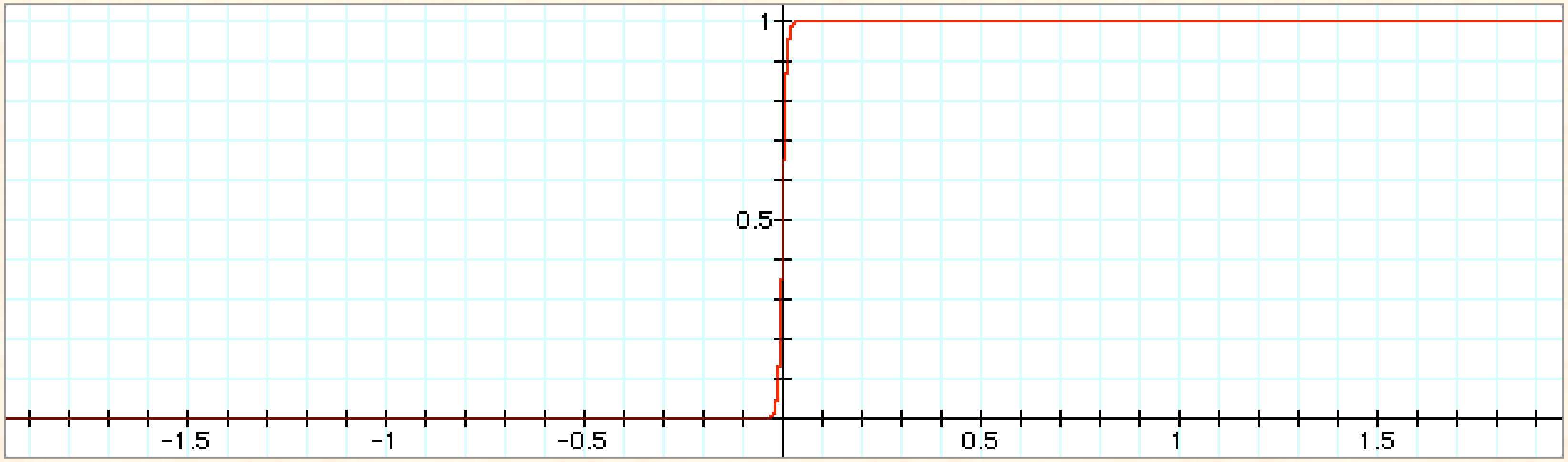


Logistic Sigmoid
$T = 0.5$

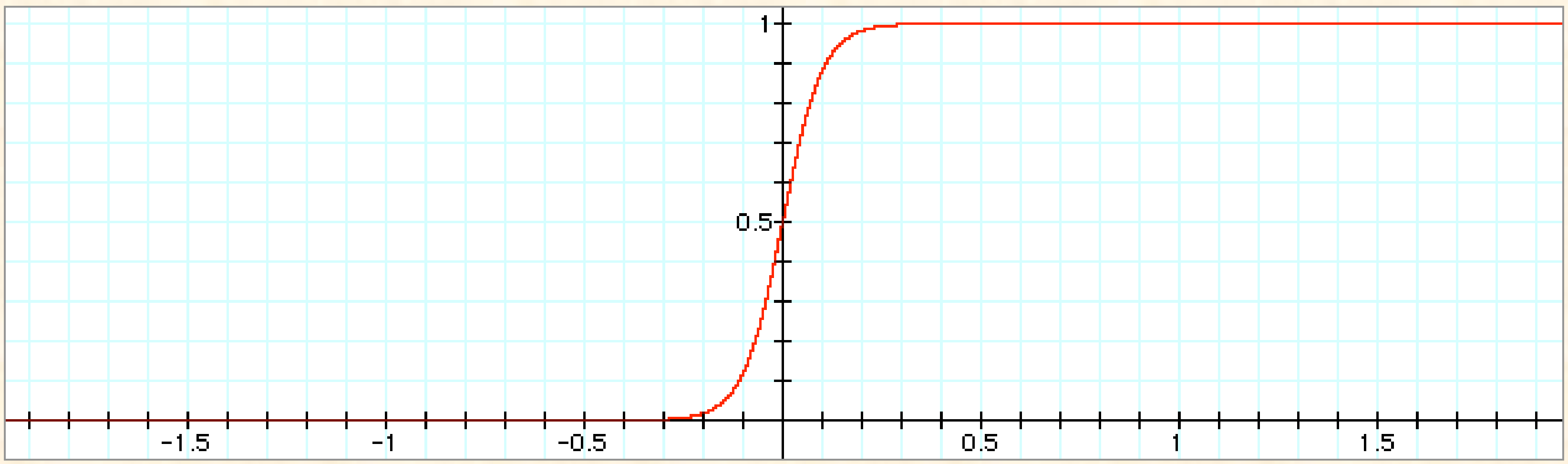


Slope at origin = 1 / 2*T*

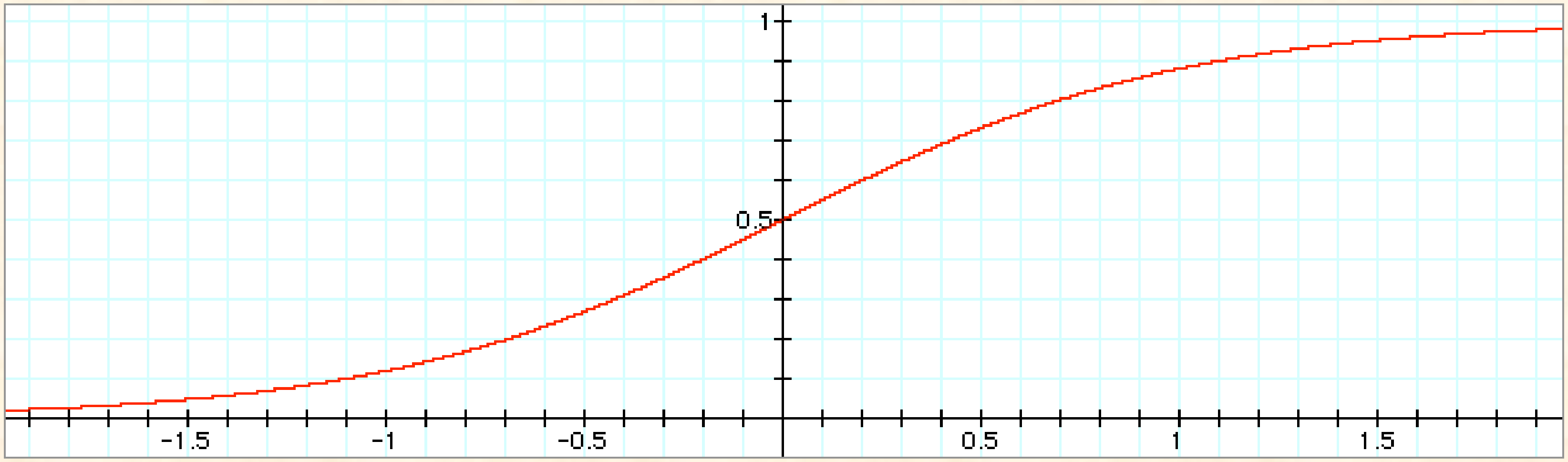
Logistic Sigmoid
$T = 0.01$



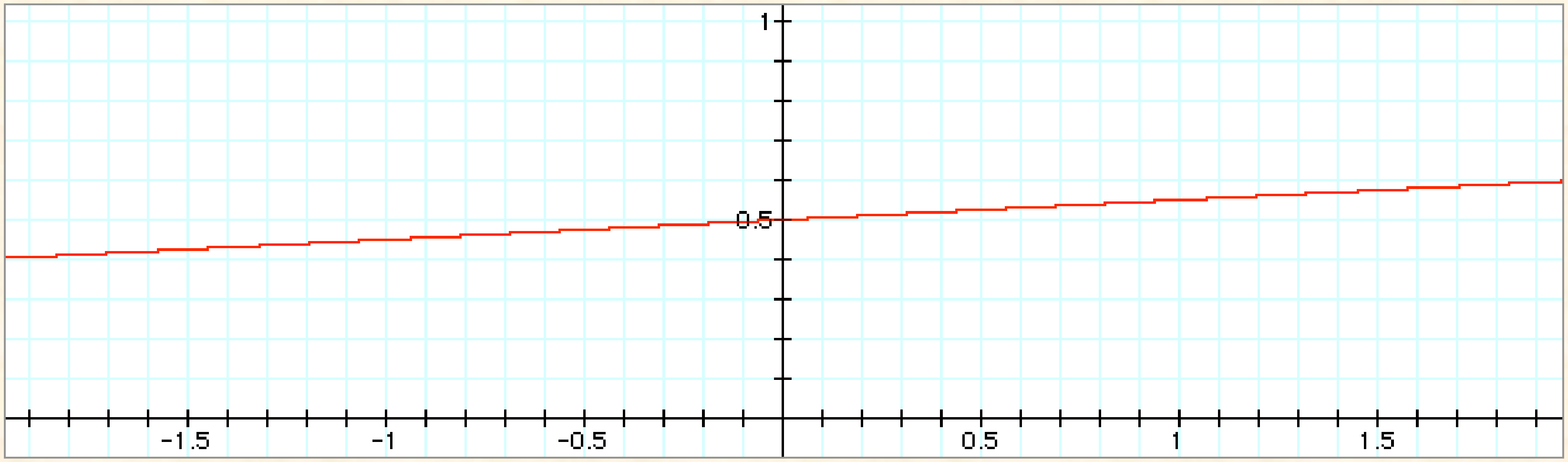
Logistic Sigmoid
$T = 0.1$



Logistic Sigmoid
$T = 1$



Logistic Sigmoid
$T = 10$



Logistic Sigmoid
$T = 100$