Homework 1 — CS 594 only! — Due: Sept. 29, 2004

General Instructions

- Please work on this assignment independently.
- You can hand in your assignment electronically (email to ytang@cs.utk.edu) or as hardcopy (put it in Yifan Tang's mailbox).

Introduction

In class (Lecture 7) we studied an activation-inhibition pattern generation system defined by a CA with the update rule:

$$s_i(t+1) = \operatorname{sign}\left[h + J_1 \sum_{r_{ij} < R_1} s_j(t) + J_2 \sum_{R_1 < r_{ij} < R} s_j(t)\right].$$

Recall that r_{ij} represents the distance between cells i and j, so the first summation is over all cells within a distance of R_1 to cell i, and the second summation is over all cells with a distance between R_1 and R_2. (It doesn't matter what you do with the $r_{ij} = R_1$ case; include it in the first or second summation as you like.) For simplicity, assume that the R_1 neighborhood does *not* include the center cell i.

The state of a CA can be updated either *synchronously* or *asynchronously*. With synchronous updating, which is what we usually do, all the states are updated simultaneously. With asynchronous updating the cells are updated one at a time (usually in some random order).

This homework assignment explores the stability of this activation-inhibition system; that is, does it inevitably reach a stable state?

Problems

Problem 1

Prove that if the states are updated asynchronously, then the CA must reach a stable state.

Hint: Define the following function (called an *energy* or *Lyapunov* function) of the total state of a CA:[1]

$$E\{\mathbf{s}(t)\} = -\tfrac{1}{2}\sum_{i} s_i(t)\left[2h + J_1 \sum_{r_{ij}<R_1} s_j(t) + J_2 \sum_{R_1<r_{ij}<R} s_j(t)\right].$$

Show that updating any single cell, according to the state update rule, cannot increase this function (that is, $\Delta E \leq 0$). What else do you need to show in order to guarantee convergence to a stable state?

Extra Credit: Assume that the R_1 neighborhood *does* include the center cell, and explore any additional assumptions that might be needed to guarantee convergence.

Problem 2

Prove, by exhibiting a counter-example, that if synchronous updating is used, then the CA may not reach a stable state.

Hint: Construct a very simple CA, obeying the above state update equation, that cycles between two different states.

[1] For this energy function, look in Bar-Yam on p. 630 (section 7.2.2) and p. 170 (sec. 1.6.6).