IV. Cooperation & Competition

Game Theory and the Iterated Prisoner's Dilemma

The Rudiments of Game Theory

Leibniz on Game Theory

- "Games combining chance and skill give the best representation of human life, particularly of military affairs and of the practice of medicine which necessarily depend partly on skill and partly on chance." — Leibniz (1710)
- "… it would be desirable to have a complete study made of games, treated mathematically." — Leibniz (1715)

Origins of Modern Theory

- 1928: John von Neumann: optimal strategy for two-person zero-sum games
  - von Neumann: mathematician & pioneer computer scientist (CAs, "von Neumann machine")
- 1944: von Neumann & Oskar Morgenstern: *Theory of Games and Economic Behavior*
  - Morgenstern: famous mathematical economist
- 1950: John Nash: *Non-cooperative Games*
  - his PhD dissertation (27 pages)
  - "genius," Nobel laureate (1994), schizophrenic

Classification of Games

- Games of Chance
  - outcome is independent of players' actions
  - "uninteresting" (apply probability theory)
- Games of Strategy
  - outcome is at least partially dependent on players' actions
  - completely in chess
  - partially in poker

Classification of Strategy Games

- Number of players (1, 2, 3, …, n)
- Zero-sum or non zero-sum
- Essential or inessential
- Perfect or imperfect information

Zero-sum vs. Non Zero-sum

- Zero-sum: winnings of some is exactly compensated by losses of others
  - sum is zero for *every* set of strategies
- Non zero-sum:
  - positive sum (mutual gain)
  - negative sum (mutual loss)
  - constant sum
  - nonconstant sum (variable gain or loss)

Essential vs. Inessential

- Essential: there is an advantage in forming coalitions
  - may involve agreements for payoffs, cooperation, etc.
  - can happen in zero-sum games only if $n \geq 3$
- Inessential: there is no such advantage
  - "everyone for themselves"

Perfect vs. Imperfect Information

- Perfect information: everyone has complete information about all previous moves
- Imperfect information: some or all have only partial information
  - players need not have complete information even about themselves (e.g. bridge)

Strategies

- Strategy: a complete sequence of actions for a player
- Pure strategy: the plan of action is completely determined
  - for each situation, a specific action is prescribed
  - disclosing the strategy might or might not be disadvantageous
- Mixed strategy: a probability is assigned to each plan of action

Von Neumann's Solution for Two-person Zero-sum Games

Maximin Criterion

- Choose the strategy that *maximizes* the *minimum* payoff
- Also called *minimax*: minimize the maximum loss
  - since it's zero-sum, your loss is the negative of your payoff
  - pessimistic?

Example

- Two mineral water companies competing for same market
- Each has fixed cost of $5 000 (regardless of sales)
- Each company can charge $1 or $2 per bottle
  - at price of $2 can sell 5 000 bottles, earning $10 000
  - at price of $1 can sell 10 000 bottles, earning $10 000
  - if they charge same price, they split market
  - otherwise all sales are of lower priced water
  - payoff = revenue – $5 000

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Payoff Matrix

		Perrier	
		price = $1	price = $2
Apollinaris	price = $1	0, 0	5000, –5000
	price = $2	–5000, 5000	0, 0

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Maximin for A.

minimum at $1

Maximin

minimum at $2

		Perrier	
		price = $1	price = $2
Apollinaris	price = $1	0, 0	5000, –5000
	price = $2	–5000, 5000	0, 0

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Maximin for P.

		Perrier	
		price = $1	price = $2
Apollinaris	price = $1	0, 0	5000, –5000
	price = $2	–5000, 5000	0, 0

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Maximin Equilibrium

		Perrier	
		price = $1	price = $2
Apollinaris	price = $1	0, 0	5000, –5000
	price = $2	–5000, 5000	0, 0

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Implications of the Equilibrium

- If both companies act "rationally," they will pick the equilibrium prices
- If either behaves "irrationally," the other will benefit (if it acts "rationally")

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Matching Pennies

- Al and Barb each independently picks either heads or tails
- If they are both heads or both tails, Al wins
- If they are different, Barb wins

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Payoff Matrix

Minimum of each pure strategy is the same		Barb	
		head	tail
Al	head	+1, –1	–1, +1
	tail	–1, +1	+1, –1

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Mixed Strategy

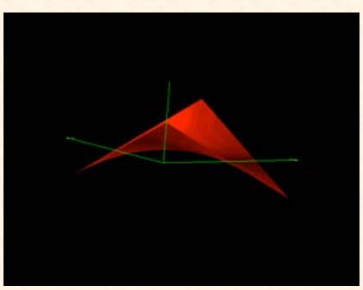
- Although we cannot use maximin to select a pure strategy, we can use it to select a mixed strategy
- Take the maximum of the minimum payoffs over all assignments of probabilities
- von Neumann proved you can always find an equilibrium if mixed strategies are permitted

Analysis

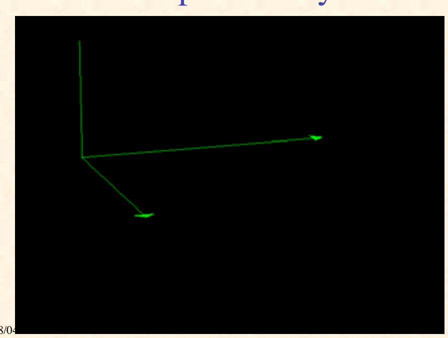
- Let P_A = probability Al picks head
- and P_B = probability Barb picks head
- Al's expected payoff:

$$E\{A\} = P_A P_B - P_A (1 - P_B) - (1 - P_A) P_B + (1 - P_A)(1 - P_B)$$
$$= (2 P_A - 1)(2 P_B - 1)$$

Al's Expected Payoff from Penny Game



How Barb's Behavior Affects Al's Expected Payoff



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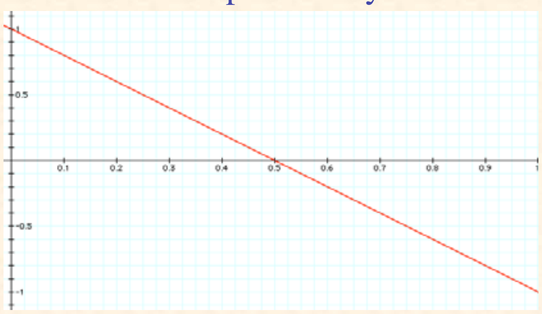

More General Analysis (Differing Payoffs)

- Let A's payoffs be:
  H = HH, h = HT, t = TH, T = TT
- $\mathrm{E}\{\mathrm{A}\} = P_A P_B H + P_A(1 - P_B)h + (1 - P_A)P_B t + (1 - P_A)(1 - P_B)T$
  $= (H + T - h - t)P_A P_B + (h - T)P_A + (t - T)P_B + T$
- To find saddle point set $\partial\mathrm{E}\{\mathrm{A}\}/\partial P_A = 0$ and $\partial\mathrm{E}\{\mathrm{A}\}/\partial P_B = 0$ to get:

$$P_A = \frac{T - t}{H + T - h - t}, \quad P_B = \frac{T - h}{H + T - h - t}$$

Random Rationality

"It seems difficult, at first, to accept the idea that 'rationality' — which appears to demand a clear, definite plan, a deterministic resolution — should be achieved by the use of probabilistic devices. Yet precisely such is the case."

—Morgenstern

Probability in Games of Chance and Strategy

- "In games of chance the task is to determine and then to evaluate probabilities inherent in the game;
- in games of strategy we *introduce* probability in order to obtain the optimal choice of strategy."

— Morgenstern

Review of von Neumann's Solution

- Every two-person zero-sum game has a maximin solution, provided we allow mixed strategies
- But— it applies only to two-person zero-sum games
- Arguably, few "games" in real life are zero-sum, except literal games (i.e., invented games for amusement)

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Nonconstant Sum Games

- There is no agreed upon definition of rationality for nonconstant sum games
- Two common criteria:
  - dominant strategy equilibrium
  - Nash equilibrium

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Dominant Strategy Equilibrium

- Dominant strategy:
  - consider each of opponents' strategies, and what your best strategy is in each situation
  - if the same strategy is best in all situations, it is the dominant strategy
- Dominant strategy equilibrium: occurs if each player has a dominant strategy and plays it

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Another Example

Price Competition		Beta		
		$p = 1$	$p = 2$	$p = 3$
Alpha	$p = 1$	0, 0	50, –10	40, –20
	$p = 2$	–10, 50	20, 20	90, 10
	$p = 3$	–20, 40	10, 90	50, 50

There is no dominant strategy

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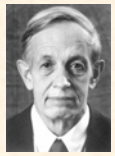
Nash Equilibrium



- Developed by John Nash in 1950
- His 27-page PhD dissertation: *Non-Cooperative Games*
- Received Nobel Prize in Economics for it in 1994
- Subject of *A Beautiful Mind*

Definition of Nash Equilibrium

- A set of strategies with the property: No player can benefit by changing actions while others keep strategies unchanged
- Players are in equilibrium if any change of strategy would lead to lower reward for that player
- For mixed strategies, we consider expected reward

Another Example (Reconsidered)

Price Competition		Beta		
		p = 1	*p* = 2	*p* = 3
Alpha	*p* = 1	0, 0	50, –10	40, –20
	p = 2	–10, 50	20, 20	90, 10
	p = 3	–20, 40	10, 90	50, 50

better for Beta better for Alpha

Not a Nash equilibrium

The Nash Equilibrium

Price Competition		Beta		
		p = 1	*p* = 2	*p* = 3
Alpha	*p* = 1	0, 0	50, –10	40, –20
	p = 2	–10, 50	20, 20	90, 10
	p = 3	–20, 40	10, 90	50, 50

Nash equilibrium

Cooperation Better for Both: A Dilemma

Price Competition		Beta		
		$p = 1$	$p = 2$	$p = 3$
Alpha	$p = 1$	0, 0	50, –10	40, –20
	$p = 2$	–10, 50	20, 20	90, 10
	$p = 3$	–20, 40	10, 90	50, 50

Cooperation