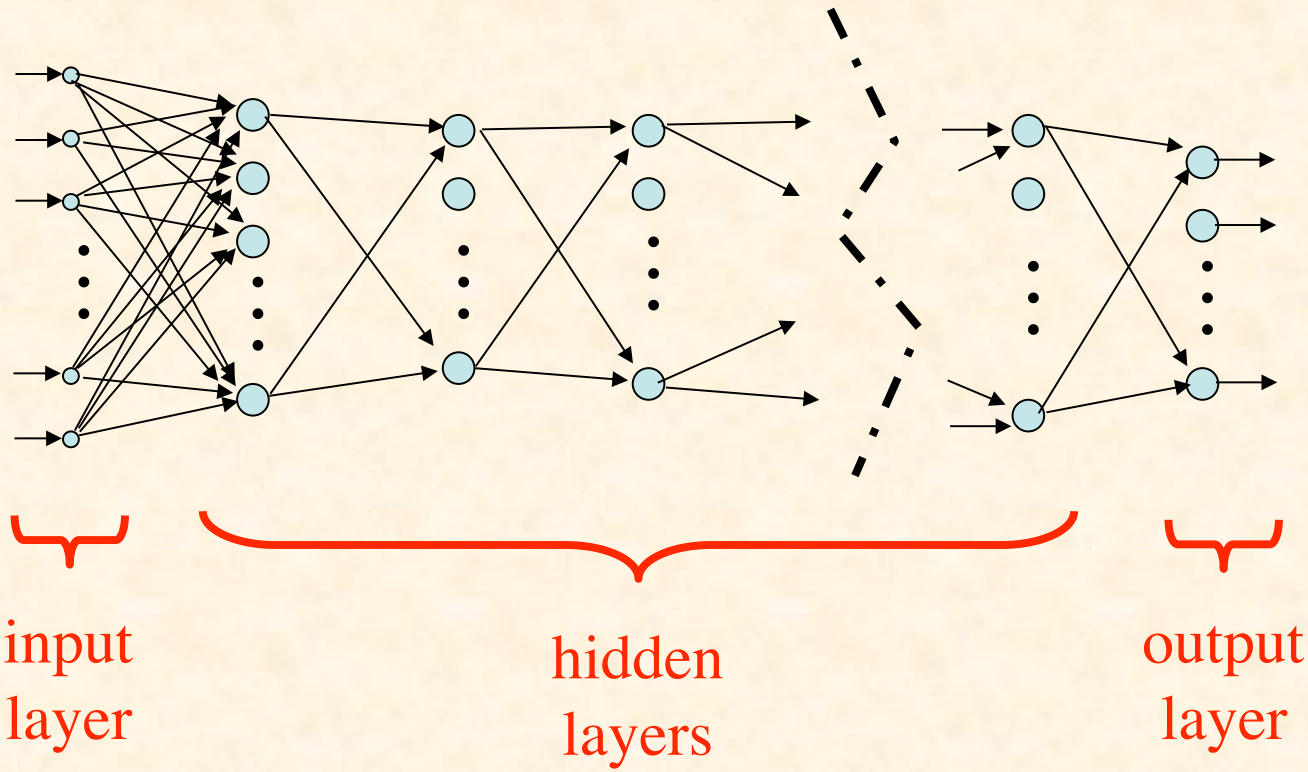
IV. Neural Network Learning

A.
Neural Network Learning

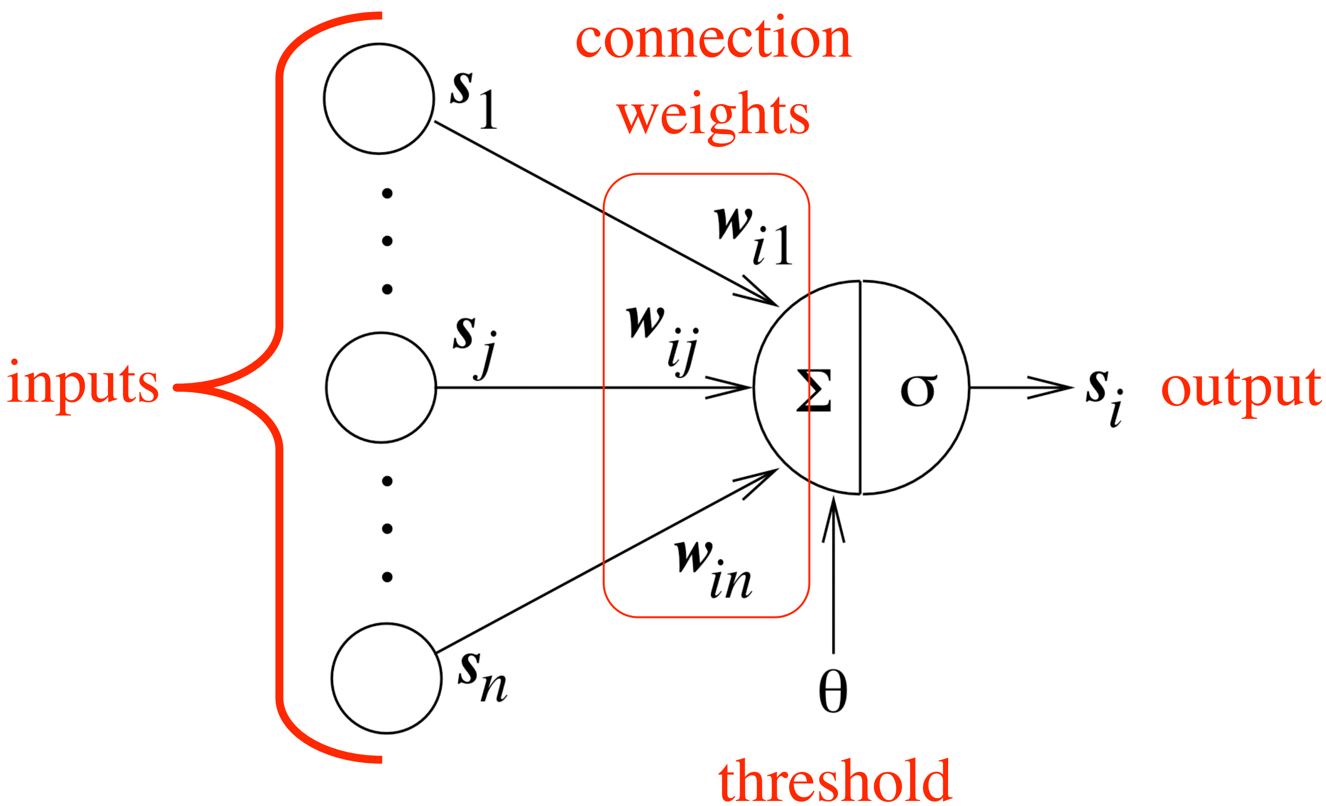
Supervised Learning

- Produce desired outputs for training inputs
- Generalize reasonably & appropriately to other inputs
- Good example: pattern recognition
- Feedforward multilayer networks

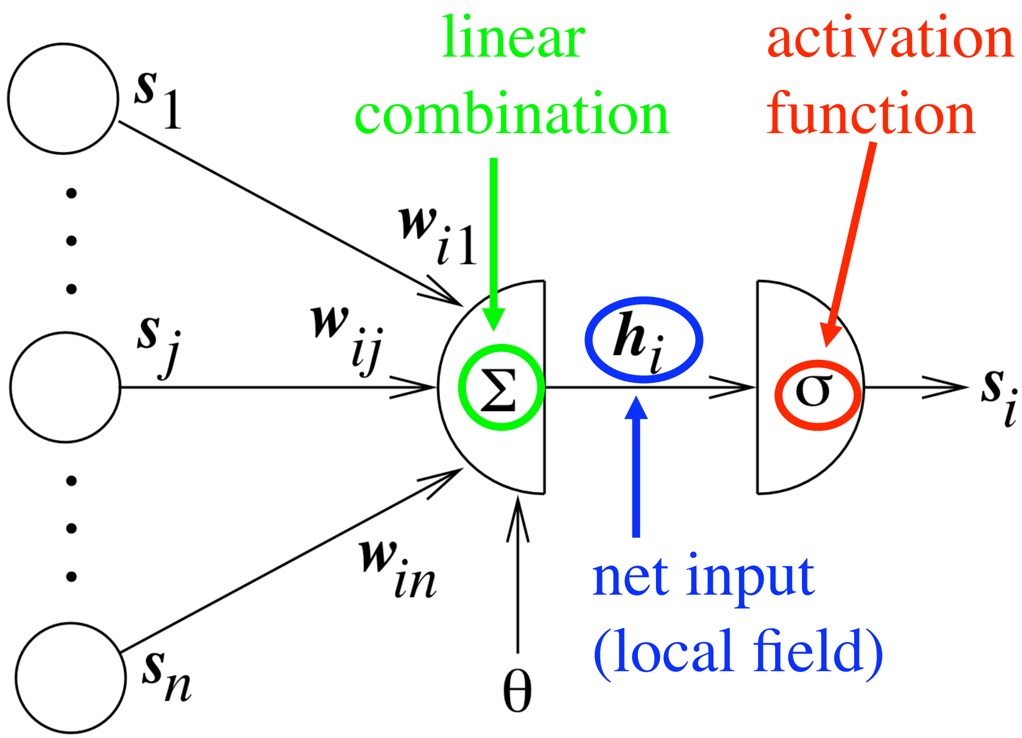
Feedforward Network



Typical Artificial Neuron



Typical Artificial Neuron



Equations

Net input:

$$h_i = \left(\sum_{j=1}^{n} w_{ij} s_j \right) - \theta$$

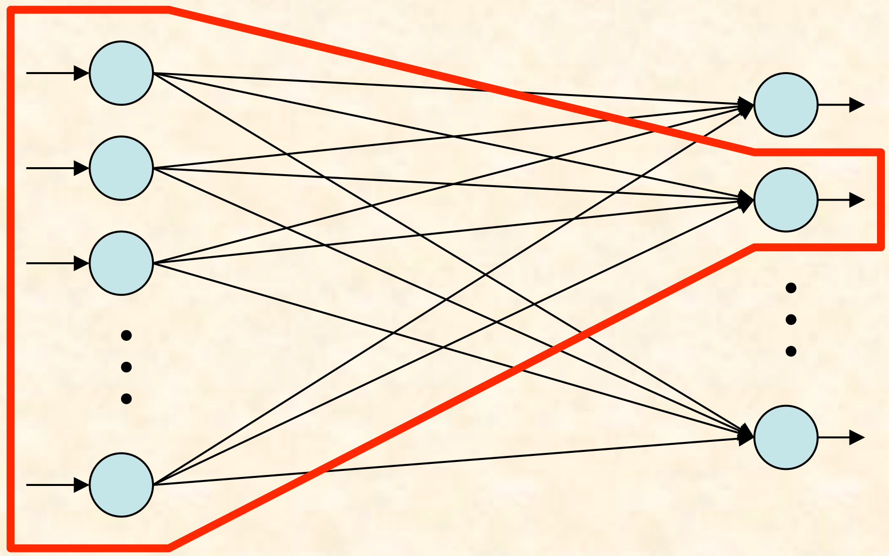
$$\mathbf{h} = \mathbf{W}\mathbf{s} - \theta$$

Neuron output:

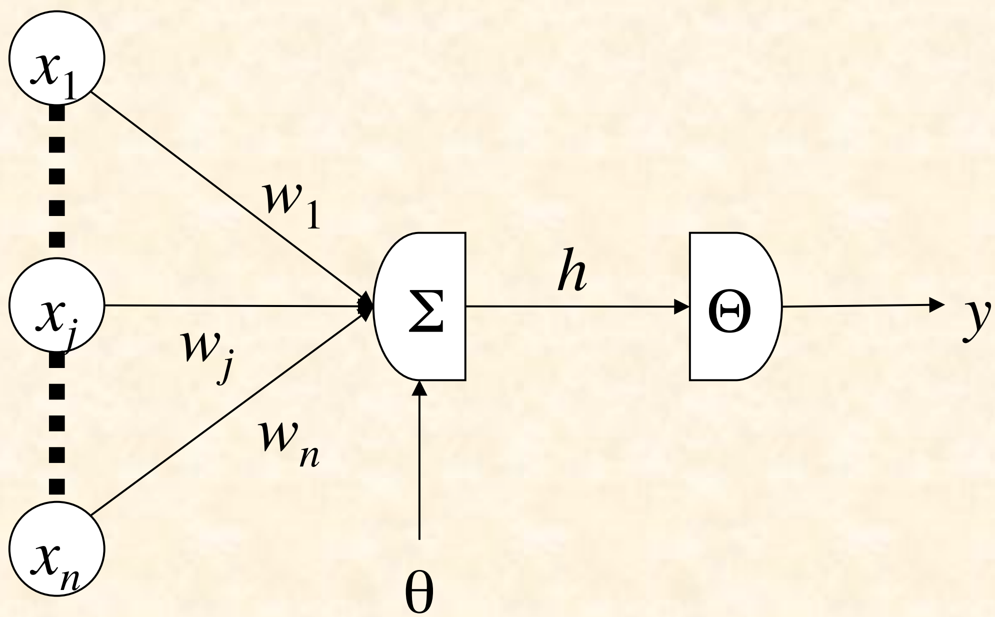
$$s_i' = \sigma(h_i)$$

$$\mathbf{s}' = \sigma(\mathbf{h})$$

Single-Layer Perceptron



Variables



Single Layer Perceptron Equations

Binary threshold activation function:

$$\sigma(h) = \Theta(h) = \begin{cases} 1, & \text{if } h > 0 \\ 0, & \text{if } h \le 0 \end{cases}$$

$$\text{Hence, } y = \begin{cases} 1, & \text{if } \sum_j w_j x_j > \theta \\ 0, & \text{otherwise} \end{cases}$$

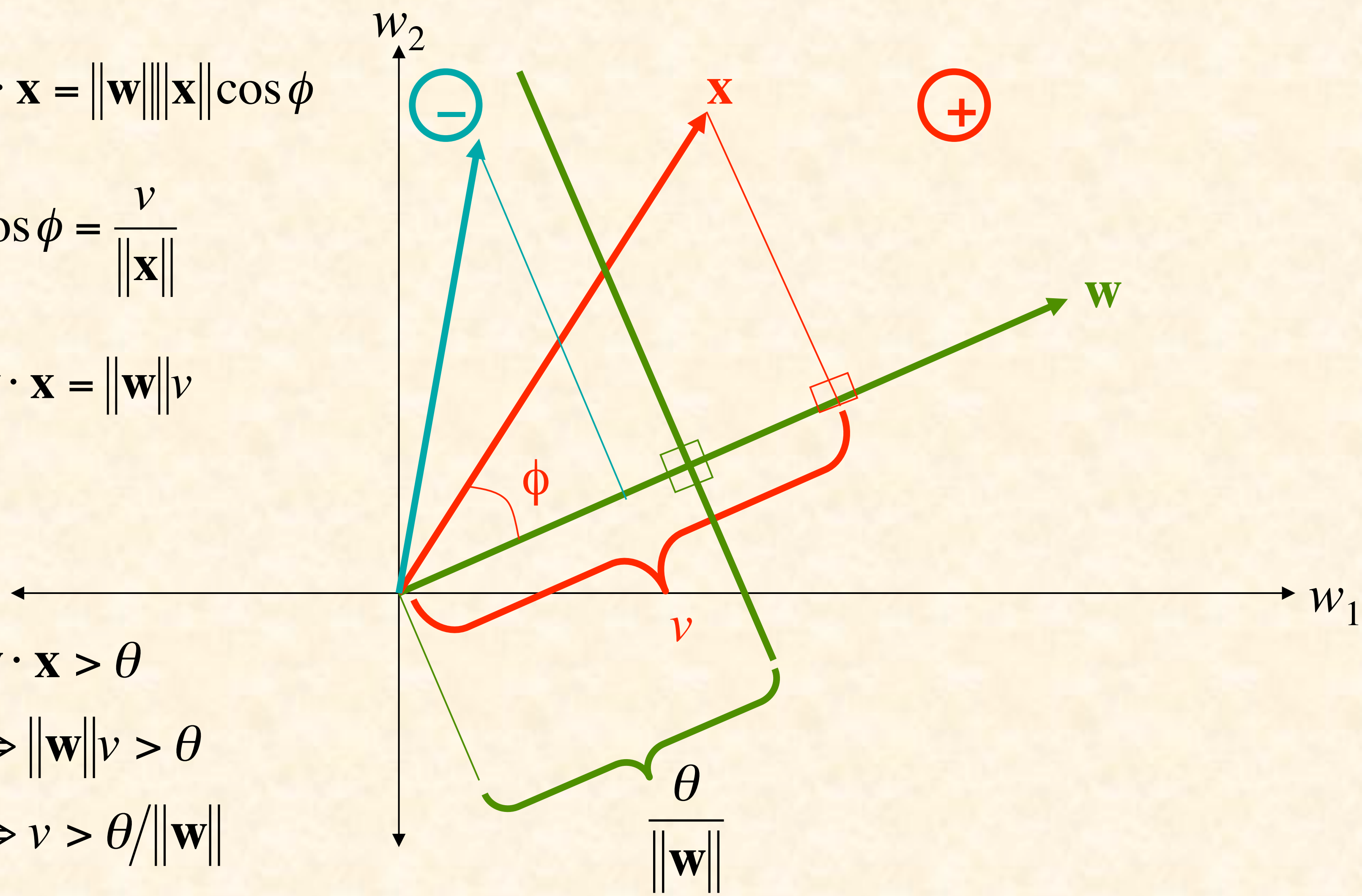
$$= \begin{cases} 1, & \text{if } \mathbf{w} \cdot \mathbf{x} > \theta \\ 0, & \text{if } \mathbf{w} \cdot \mathbf{x} \le \theta \end{cases}$$

2D Weight Vector

$$\mathbf{w} \cdot \mathbf{x} = \|\mathbf{w}\| \|\mathbf{x}\| \cos\phi$$

$$\cos\phi = \frac{v}{\|\mathbf{x}\|}$$

$$\mathbf{w} \cdot \mathbf{x} = \|\mathbf{w}\| v$$

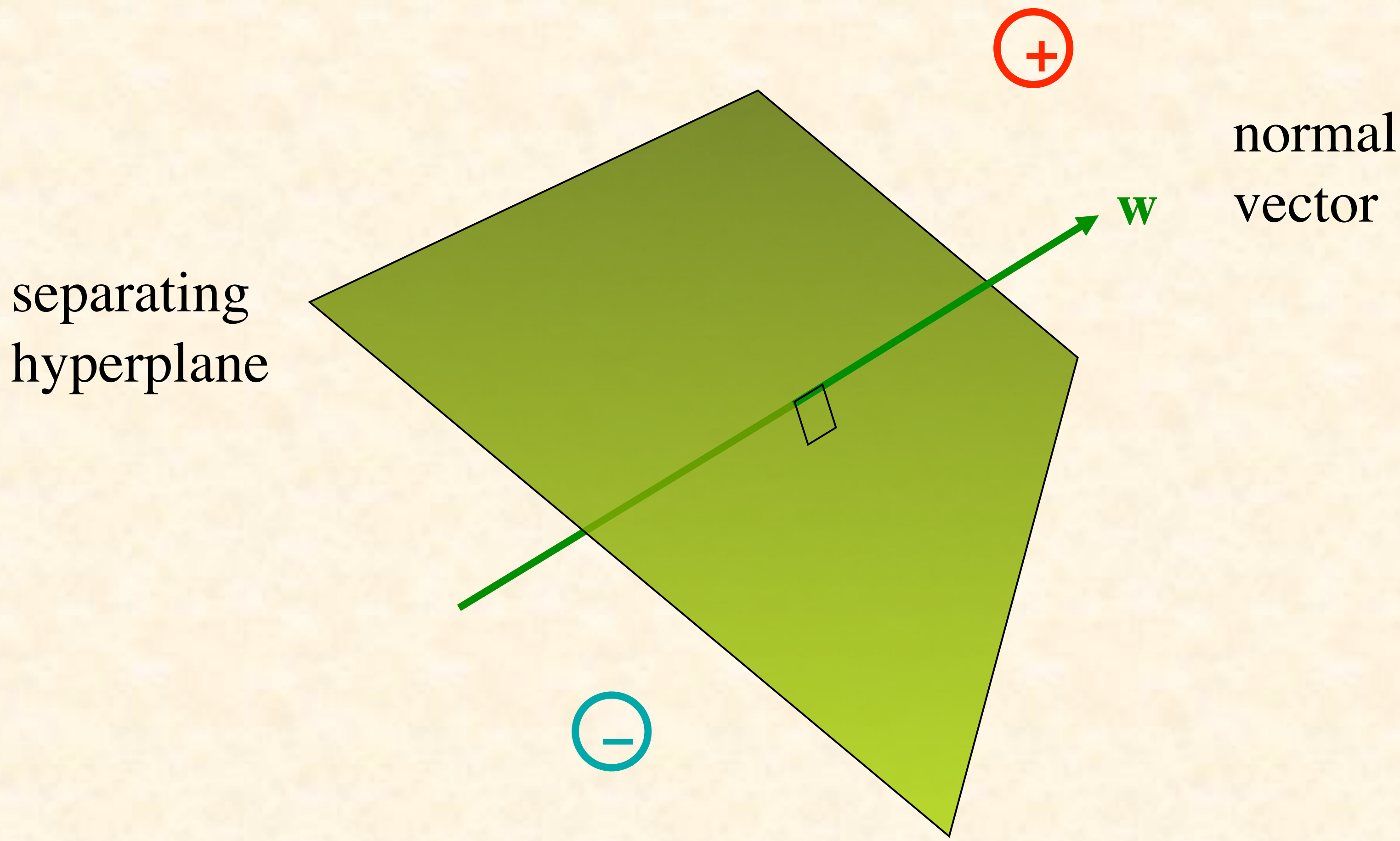


$$\mathbf{w} \cdot \mathbf{x} > \theta$$

$$\Leftrightarrow \|\mathbf{w}\| v > \theta$$

$$\Leftrightarrow v > \theta / \|\mathbf{w}\|$$

N-Dimensional Weight Vector



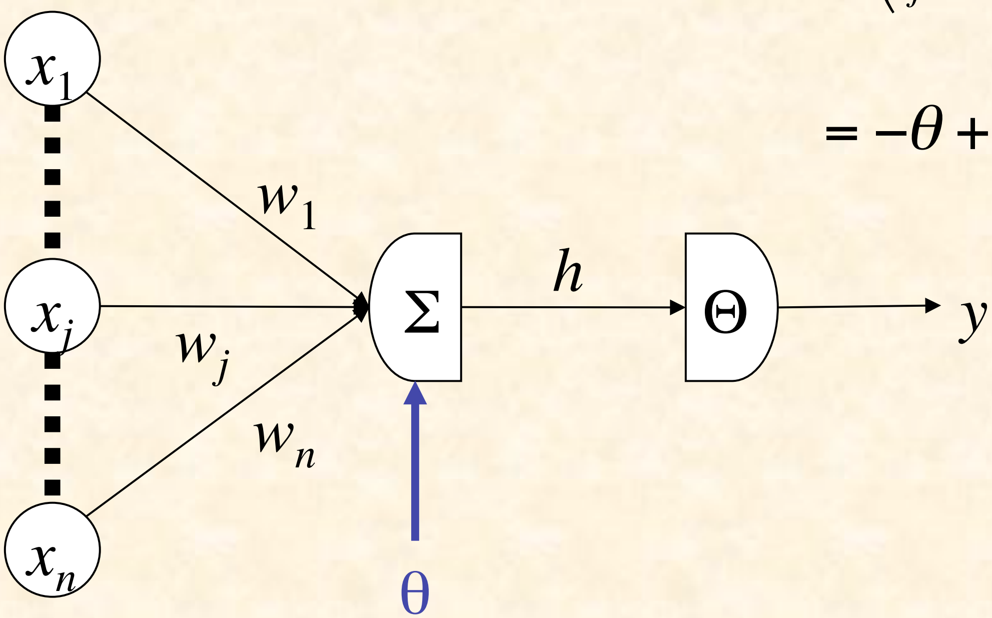
Goal of Perceptron Learning

- Suppose we have training patterns $\mathbf{x}^1$, $\mathbf{x}^2$, …, $\mathbf{x}^P$ with corresponding desired outputs $y^1, y^2, \ldots, y^P$
- where $\mathbf{x}^p \in \{0, 1\}^n$, $y^p \in \{0, 1\}$
- We want to find $\mathbf{w}$, θ such that $y^p = \Theta(\mathbf{w} \cdot \mathbf{x}^p - \theta)$ for $p = 1, \ldots, P$

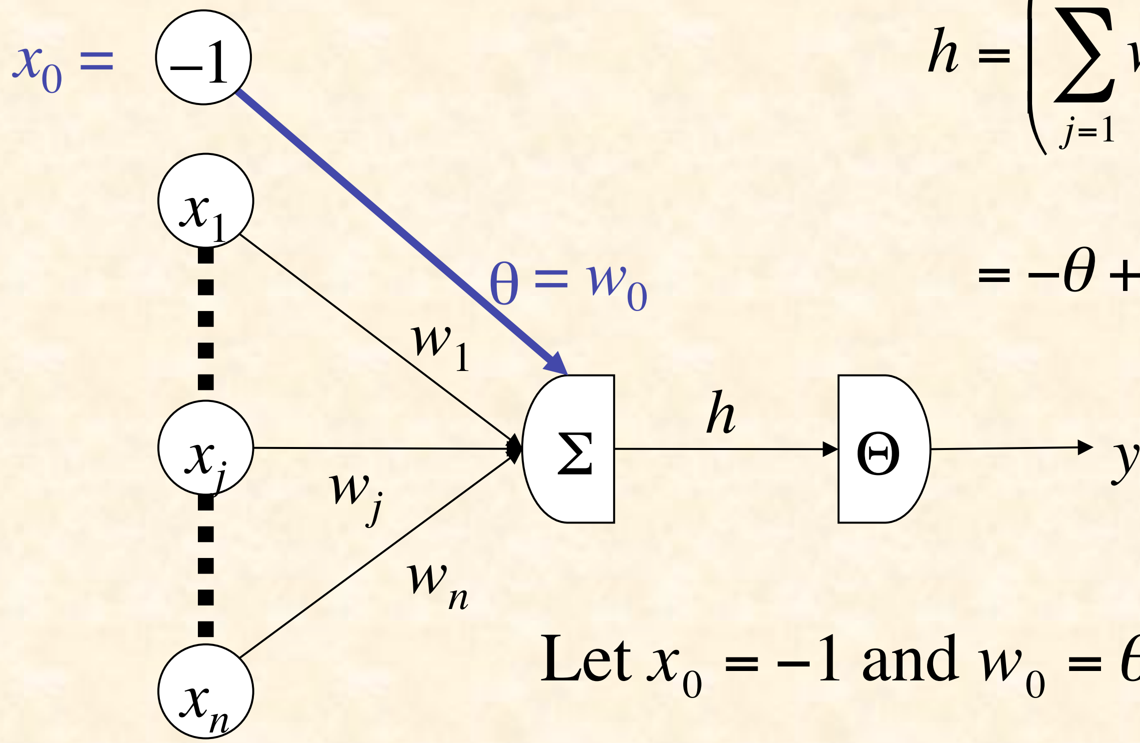
Treating Threshold as Weight

$$h = \left(\sum_{j=1}^{n} w_j x_j \right) - \theta$$

$$= -\theta + \sum_{j=1}^{n} w_j x_j$$



Treating Threshold as Weight



$$h = \left(\sum_{j=1}^{n} w_j x_j \right) - \theta$$

$$= -\theta + \sum_{j=1}^{n} w_j x_j$$

Let $x_0 = -1$ and $w_0 = \theta$

$$h = w_0 x_0 + \sum_{j=1}^{n} w_j x_j = \sum_{j=0}^{n} w_j x_j = \tilde{\mathbf{w}} \cdot \tilde{\mathbf{x}}$$

Augmented Vectors

$$\tilde{\mathbf{w}} = \begin{pmatrix} \theta \\ w_1 \\ \vdots \\ w_n \end{pmatrix} \qquad \tilde{\mathbf{x}}^p = \begin{pmatrix} -1 \\ x_1^p \\ \vdots \\ x_n^p \end{pmatrix}$$

We want $y^p = \Theta\left(\tilde{\mathbf{w}} \cdot \tilde{\mathbf{x}}^p\right), \; p = 1, \ldots, P$

Reformulation as Positive Examples

We have positive ($y^p = 1$) and negative ($y^p = 0$) examples

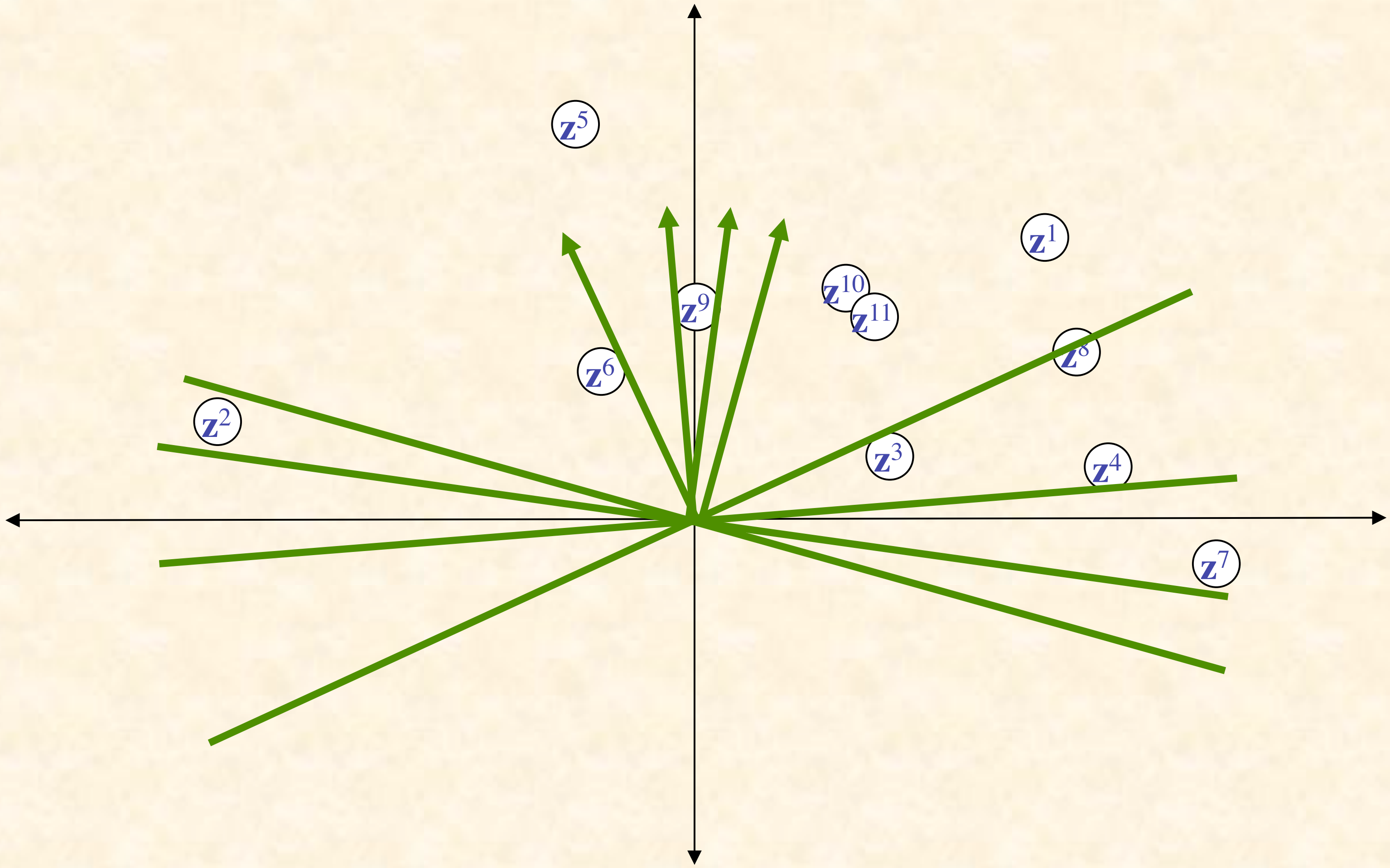
Want $\tilde{\mathbf{w}} \cdot \tilde{\mathbf{x}}^p > 0$ for positive, $\tilde{\mathbf{w}} \cdot \tilde{\mathbf{x}}^p \le 0$ for negative

Let $\mathbf{z}^p = \tilde{\mathbf{x}}^p$ for positive, $\mathbf{z}^p = -\tilde{\mathbf{x}}^p$ for negative

Want $\tilde{\mathbf{w}} \cdot \mathbf{z}^p \ge 0$, for $p = 1, \ldots, P$

Hyperplane through origin with all $\mathbf{z}^p$ on one side

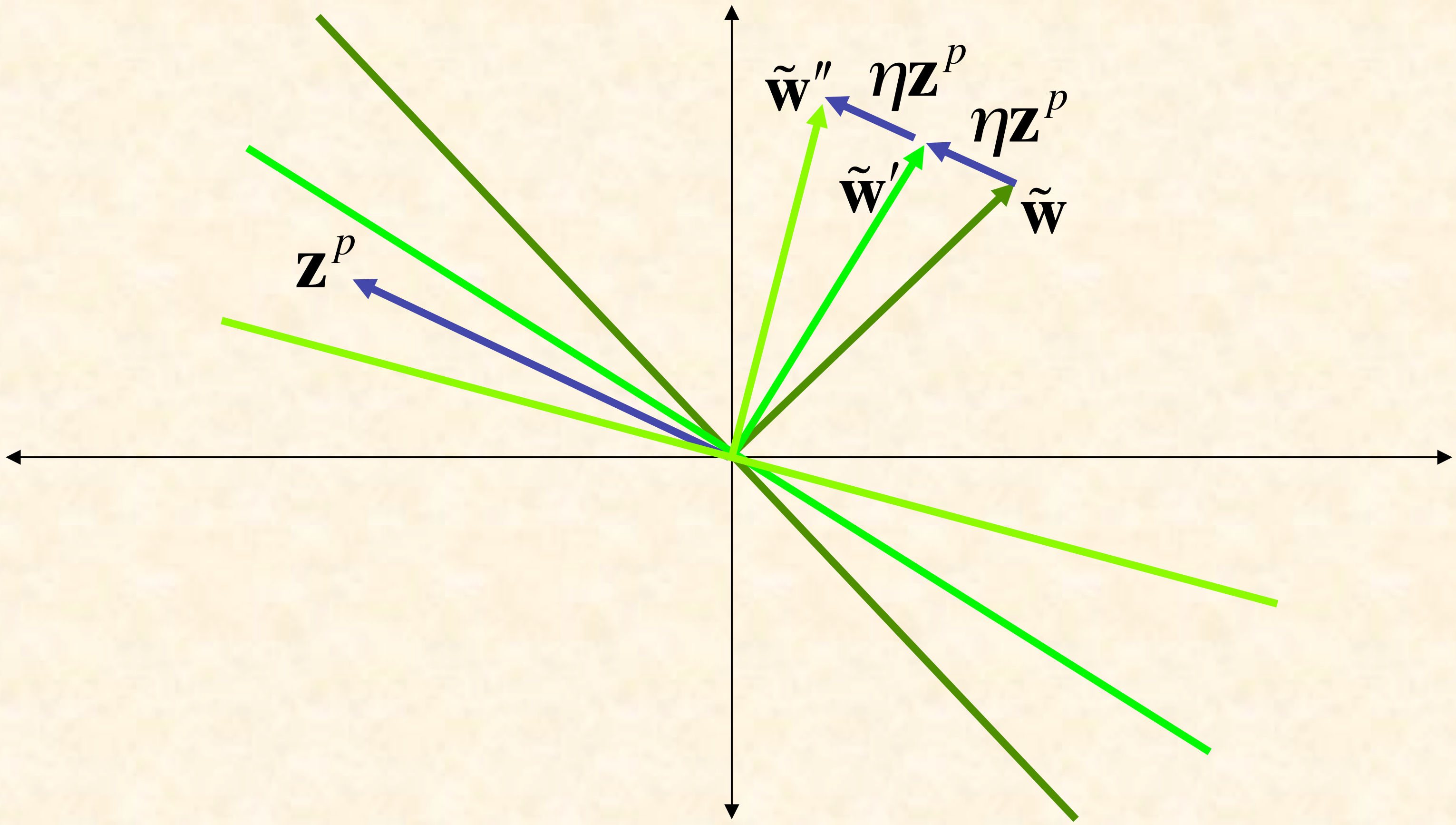
Adjustment of Weight Vector



Outline of Perceptron Learning Algorithm

1. initialize weight vector randomly
2. until all patterns classified correctly, do:
   a) for $p = 1, \ldots, P$ do:
      1) if $\mathbf{z}^p$ classified correctly, do nothing
      2) else adjust weight vector to be closer to correct classification

Weight Adjustment



Improvement in Performance

If $\tilde{\mathbf{w}} \cdot \mathbf{z}^p < 0$,

$$
\begin{aligned}
\tilde{\mathbf{w}}' \cdot \mathbf{z}^p &= \left(\tilde{\mathbf{w}} + \eta \mathbf{z}^p\right) \cdot \mathbf{z}^p \\
&= \tilde{\mathbf{w}} \cdot \mathbf{z}^p + \eta \mathbf{z}^p \cdot \mathbf{z}^p \\
&= \tilde{\mathbf{w}} \cdot \mathbf{z}^p + \eta \left\|\mathbf{z}^p\right\|^2 \\
&> \tilde{\mathbf{w}} \cdot \mathbf{z}^p
\end{aligned}
$$

Perceptron Learning Theorem

- If there is a set of weights that will solve the problem,
- then the PLA will eventually find it
- (for a sufficiently small learning rate)
- Note: only applies if positive & negative examples are linearly separable

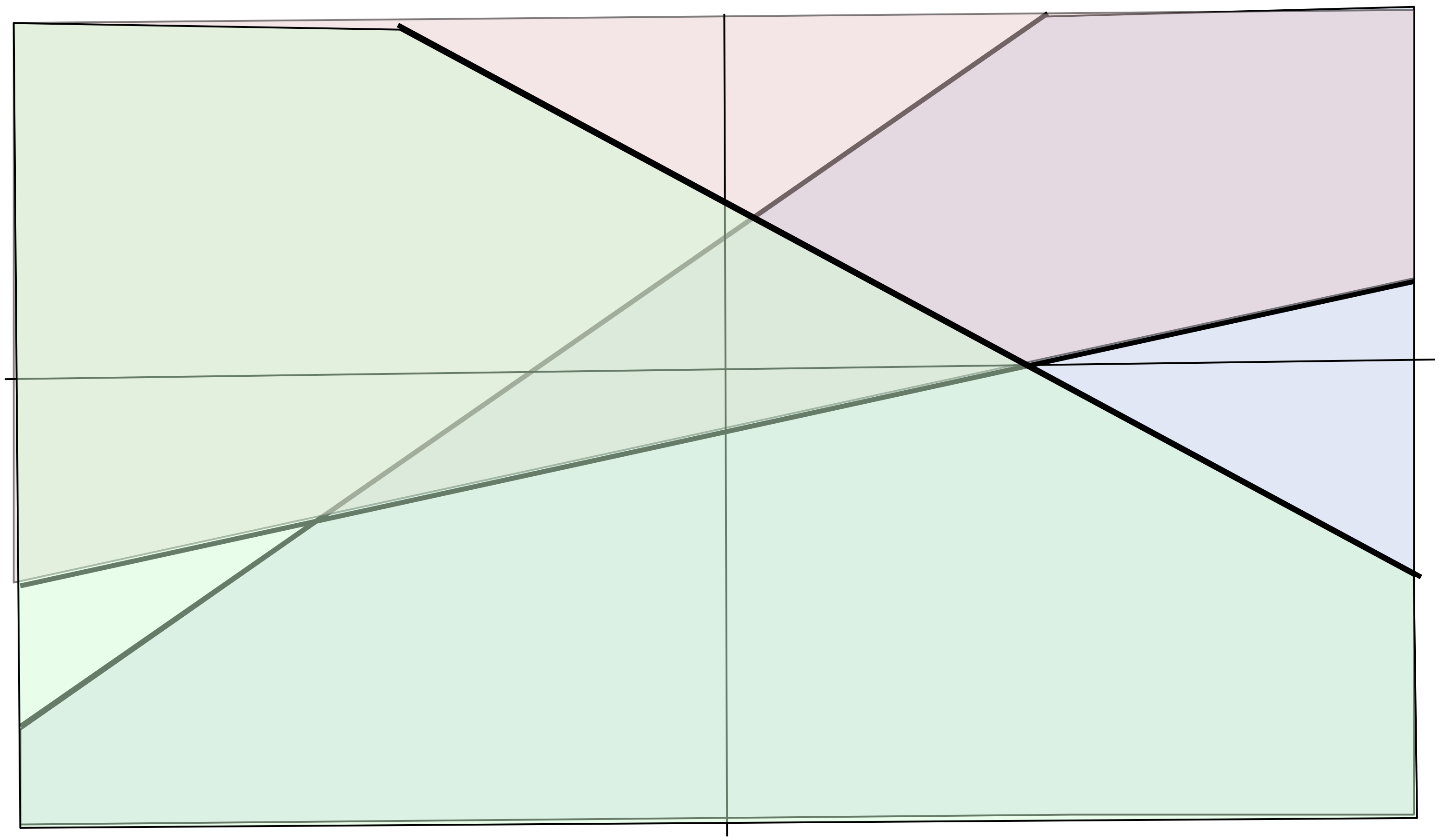
NetLogo Simulation of Perceptron Learning

Run Perceptron-Geometry.nlogo

Classification Power of Multilayer Perceptrons

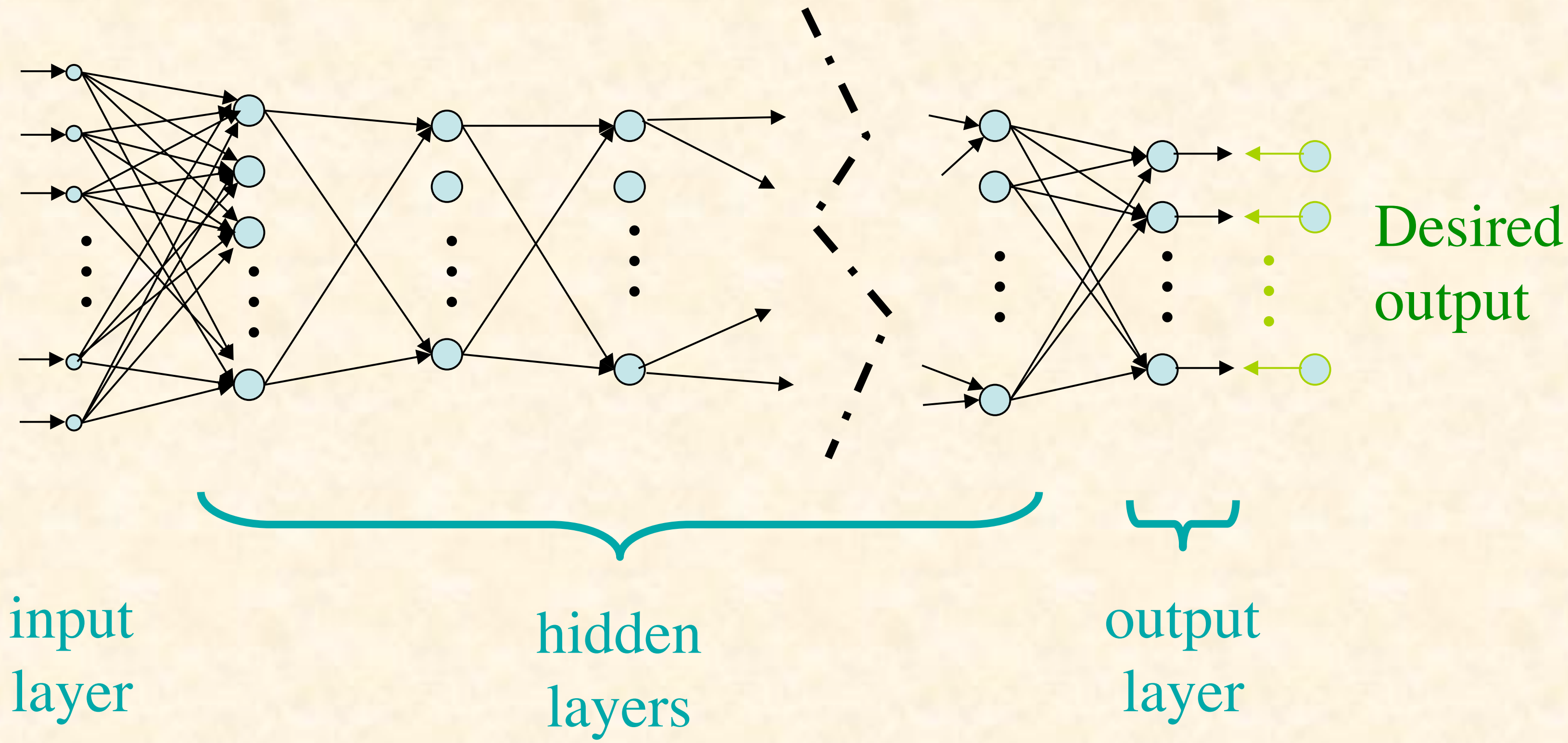
- Perceptrons can function as logic gates
- Therefore MLP can form intersections, unions, differences of linearly-separable regions
- Classes can be arbitrary *hyperpolyhedra*
- Minsky & Papert criticism of perceptrons
- No one succeeded in developing a MLP learning algorithm

Hyperpolyhedral Classes



Credit Assignment Problem

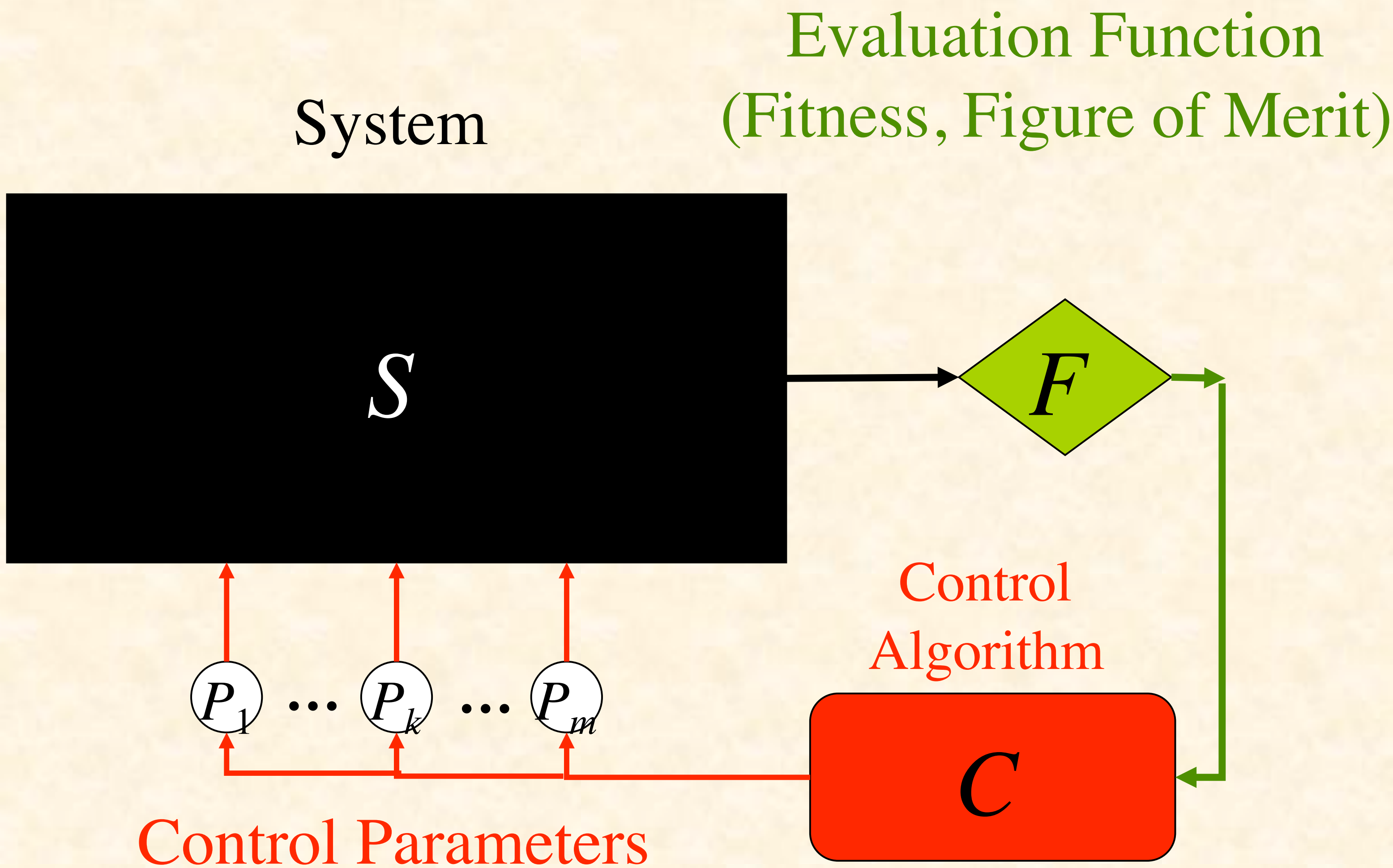
How do we adjust the weights of the hidden layers?



NetLogo Demonstration of Back-Propagation Learning

Run Artificial Neural Net.nlogo

Adaptive System



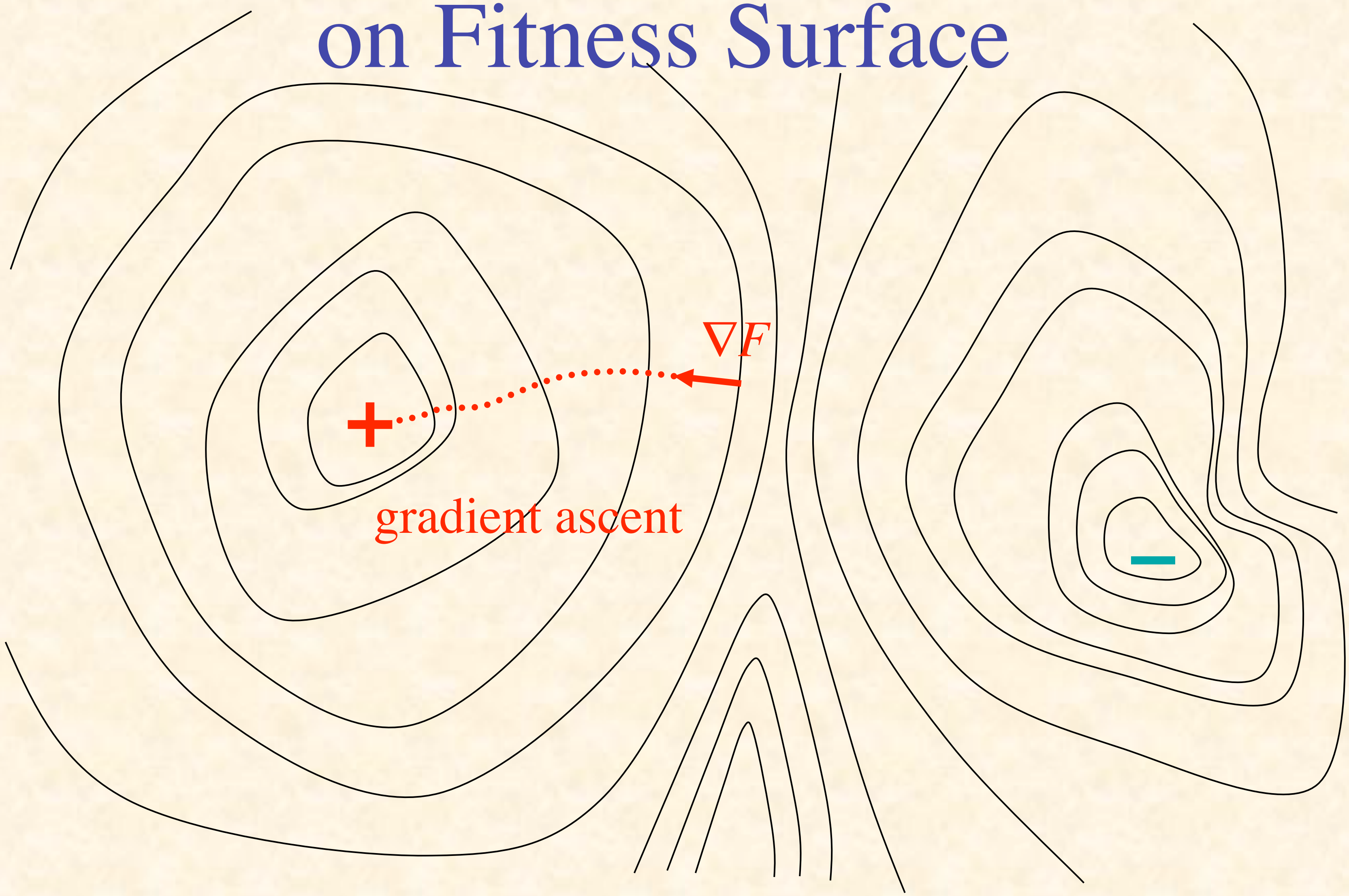
Gradient

$\frac{\partial F}{\partial P_k}$ measures how F is altered by variation of P_k

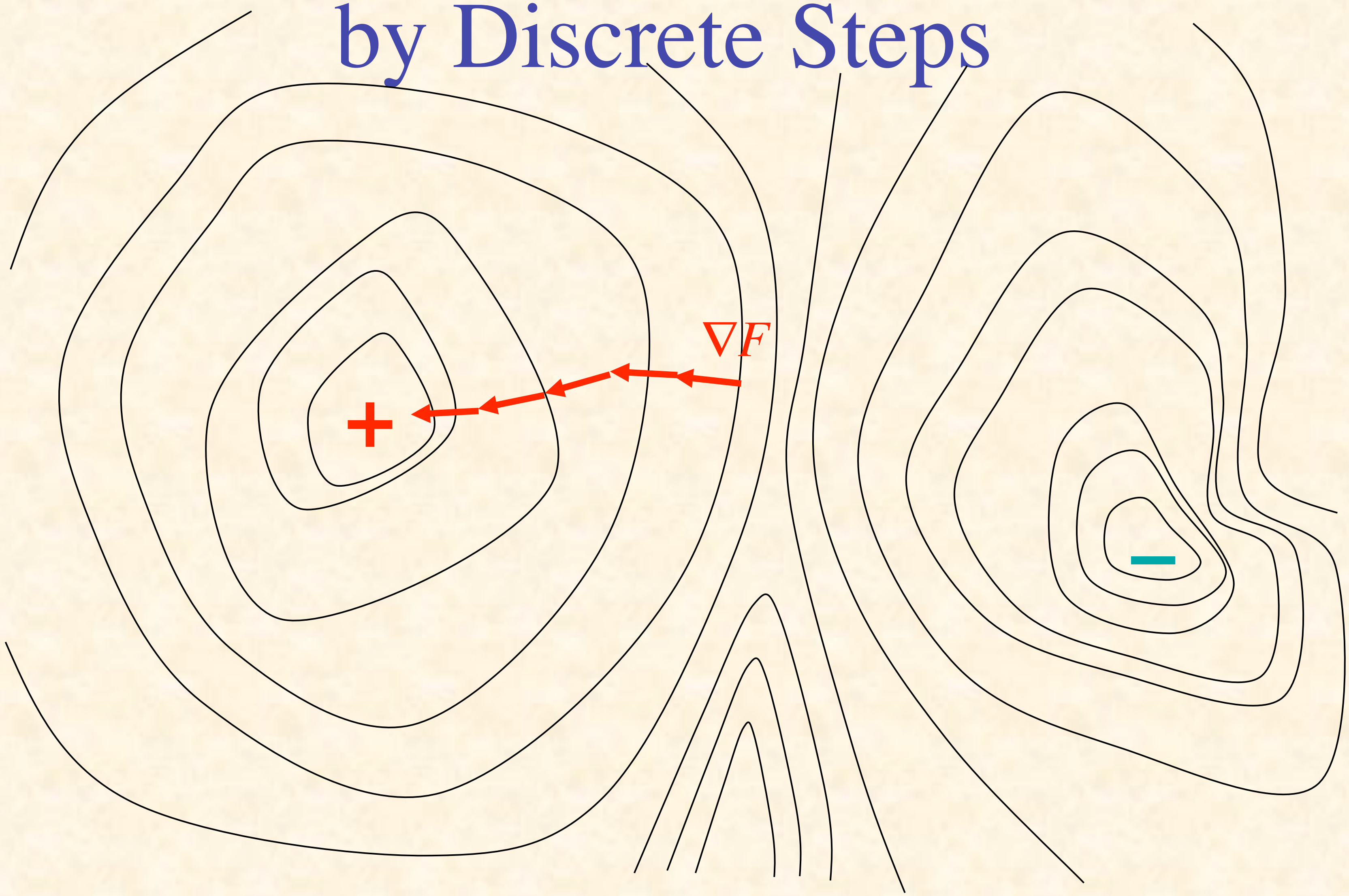
$$\nabla F = \begin{pmatrix} \partial F / \partial P_1 \\ \vdots \\ \partial F / \partial P_k \\ \vdots \\ \partial F / \partial P_m \end{pmatrix}$$

∇F points in direction of maximum local increase in F

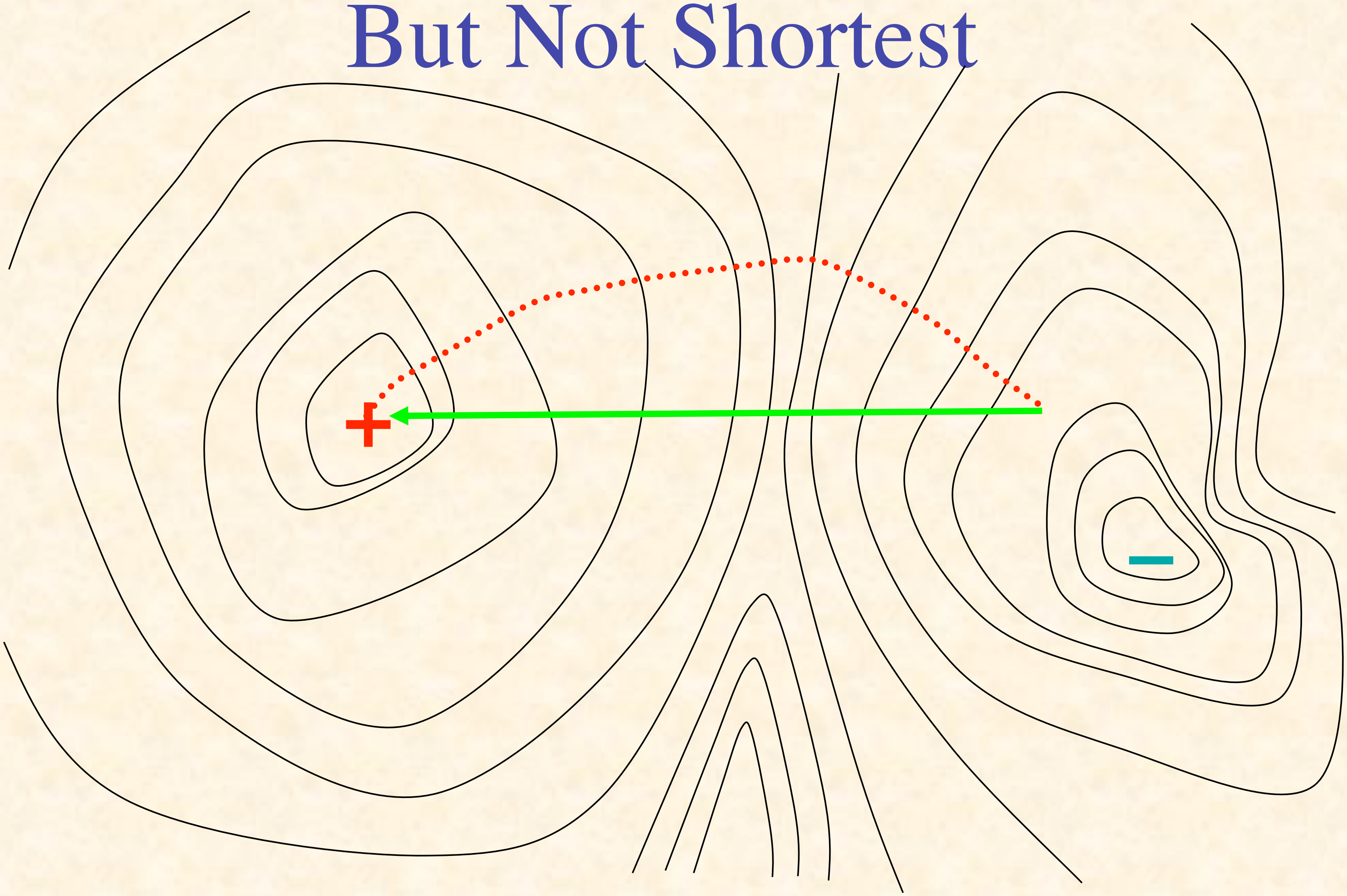
Gradient Ascent on Fitness Surface



Gradient Ascent by Discrete Steps



Gradient Ascent is Local But Not Shortest



Gradient Ascent Process

$$\dot{\mathbf{P}} = \eta \nabla F(\mathbf{P})$$

Change in fitness:

$$\dot{F} = \frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{k=1}^{m} \frac{\partial F}{\partial P_k} \frac{\mathrm{d}P_k}{\mathrm{d}t} = \sum_{k=1}^{m} (\nabla F)_k \dot{P}_k$$

$$\dot{F} = \nabla F \cdot \dot{\mathbf{P}}$$

$$\dot{F} = \nabla F \cdot \eta \nabla F = \eta \|\nabla F\|^2 \geq 0$$

Therefore gradient ascent increases fitness
(until reaches 0 gradient)

General Ascent in Fitness

Note that any adaptive process $\mathbf{P}(t)$ will increase fitness provided:

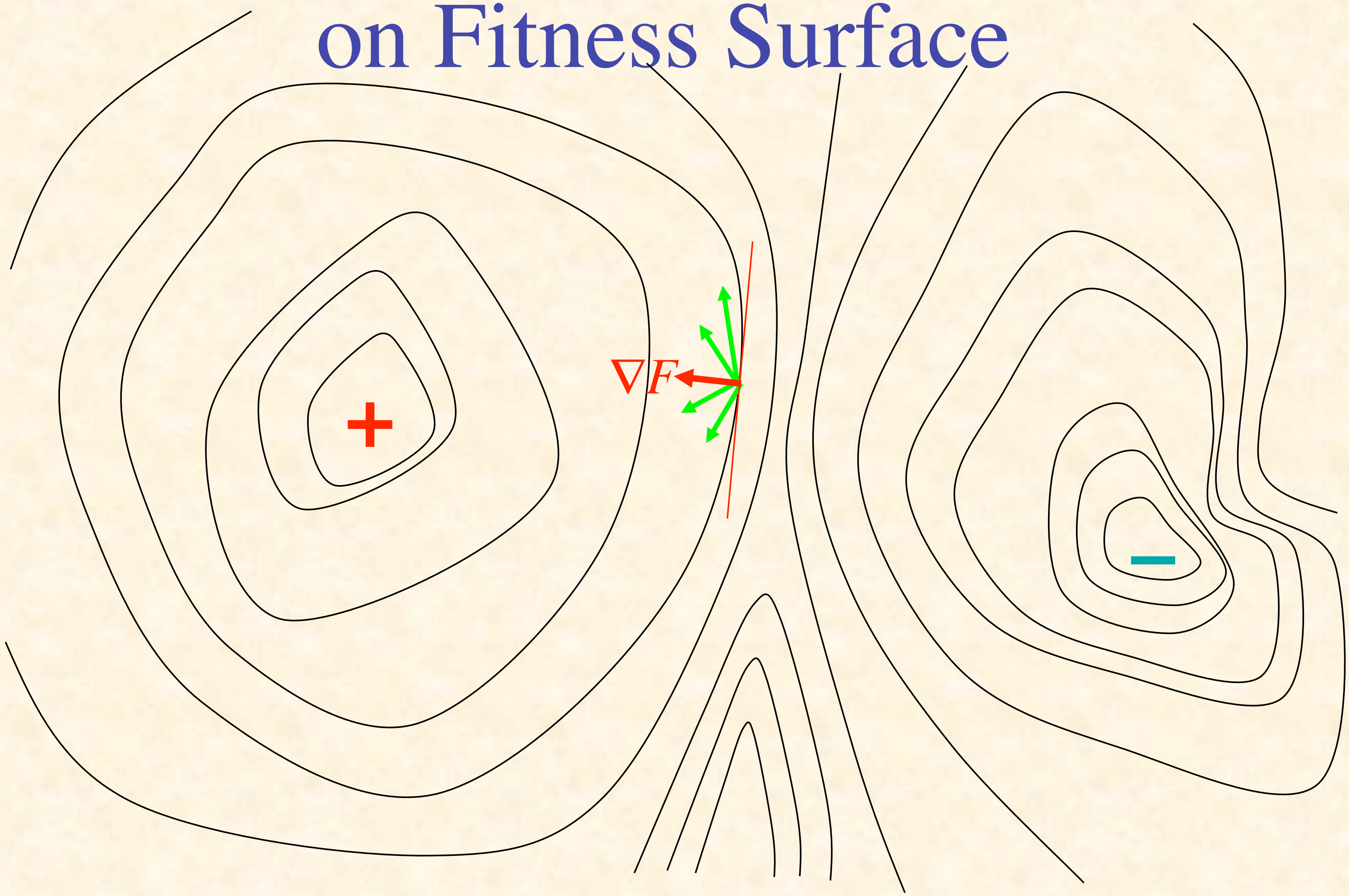
$$0 < \dot{F} = \nabla F \cdot \dot{\mathbf{P}} = \|\nabla F\| \|\dot{\mathbf{P}}\| \cos\varphi$$

where φ is angle between ∇F and $\dot{\mathbf{P}}$

Hence we need $\cos\varphi > 0$

or $|\varphi| < 90^\circ$

General Ascent on Fitness Surface



Fitness as Minimum Error

Suppose for Q different inputs we have target outputs $\mathbf{t}^1,\ldots,\mathbf{t}^Q$

Suppose for parameters $\mathbf{P}$ the corresponding actual outputs are $\mathbf{y}^1,\ldots,\mathbf{y}^Q$

Suppose $D(\mathbf{t},\mathbf{y}) \in [0,\infty)$ measures difference between target & actual outputs

Let $E^q = D(\mathbf{t}^q,\mathbf{y}^q)$ be error on qth sample

Let $F(\mathbf{P}) = -\sum_{q=1}^{Q} E^q(\mathbf{P}) = -\sum_{q=1}^{Q} D\left[\mathbf{t}^q,\mathbf{y}^q(\mathbf{P})\right]$

Gradient of Fitness

$$\nabla F = \nabla\left(-\sum_q E^q\right) = -\sum_q \nabla E^q$$

$$\frac{\partial E^q}{\partial P_k} = \frac{\partial}{\partial P_k} D\left(\mathbf{t}^q, \mathbf{y}^q\right) = \sum_j \frac{\partial D\left(\mathbf{t}^q, \mathbf{y}^q\right)}{\partial y_j^q} \frac{\partial y_j^q}{\partial P_k}$$

$$= \frac{\mathrm{d}\, D\left(\mathbf{t}^q, \mathbf{y}^q\right)}{\mathrm{d}\,\mathbf{y}^q} \cdot \frac{\partial \mathbf{y}^q}{\partial P_k}$$

$$= \nabla_{\mathbf{y}^q} D\left(\mathbf{t}^q, \mathbf{y}^q\right) \cdot \partial \mathbf{y}^q \big/ \partial P_k$$

Jacobian Matrix

Define Jacobian matrix $\mathbf{J}^q = \begin{pmatrix} \partial y_1^q / \partial P_1 & \cdots & \partial y_1^q / \partial P_m \\ \vdots & \ddots & \vdots \\ \partial y_n^q / \partial P_1 & \cdots & \partial y_n^q / \partial P_m \end{pmatrix}$

Note $\mathbf{J}^q \in \Re^{n \times m}$ and $\nabla D(\mathbf{t}^q, \mathbf{y}^q) \in \Re^{n \times 1}$

Since $\left(\nabla E^q\right)_k = \dfrac{\partial E^q}{\partial P_k} = \sum_j \dfrac{\partial y_j^q}{\partial P_k} \dfrac{\partial D(\mathbf{t}^q, \mathbf{y}^q)}{\partial y_j^q}$,

$\therefore \nabla E^q = \left(\mathbf{J}^q\right)^{\mathrm{T}} \nabla D(\mathbf{t}^q, \mathbf{y}^q)$

Derivative of Squared Euclidean Distance

Suppose $D(\mathbf{t},\mathbf{y}) = \|\mathbf{t} - \mathbf{y}\|^2 = \sum_i (t_i - y_i)^2$

$$\frac{\partial D(\mathbf{t} - \mathbf{y})}{\partial y_j} = \frac{\partial}{\partial y_j} \sum_i (t_i - y_i)^2 = \sum_i \frac{\partial (t_i - y_i)^2}{\partial y_j}$$

$$= \frac{\mathrm{d}(t_j - y_j)^2}{\mathrm{d}\, y_j} = -2(t_j - y_j)$$

$$\therefore \frac{\mathrm{d}\, D(\mathbf{t},\mathbf{y})}{\mathrm{d}\,\mathbf{y}} = 2(\mathbf{y} - \mathbf{t})$$

Gradient of Error on q^{th} Input

$$\frac{\partial E^q}{\partial P_k} = \frac{\mathrm{d}\, D\left(\mathbf{t}^q, \mathbf{y}^q\right)}{\mathrm{d}\,\mathbf{y}^{\mathbf{q}}} \cdot \frac{\partial \mathbf{y}^q}{\partial P_k}$$

$$= 2\left(\mathbf{y}^q - \mathbf{t}^q\right) \cdot \frac{\partial \mathbf{y}^q}{\partial P_k}$$

$$= 2 \sum_j \left(y_j^q - t_j^q\right) \frac{\partial y_j^q}{\partial P_k}$$

$$\nabla E^q = 2\left(\mathbf{J}^q\right)^{\mathrm{T}} \left(\mathbf{y}^q - \mathbf{t}^q\right)$$

Recap

$$\dot{\mathbf{P}} = \eta \sum_q \left(\mathbf{J}^q\right)^{\mathrm{T}} \left(\mathbf{t}^q - \mathbf{y}^q\right)$$

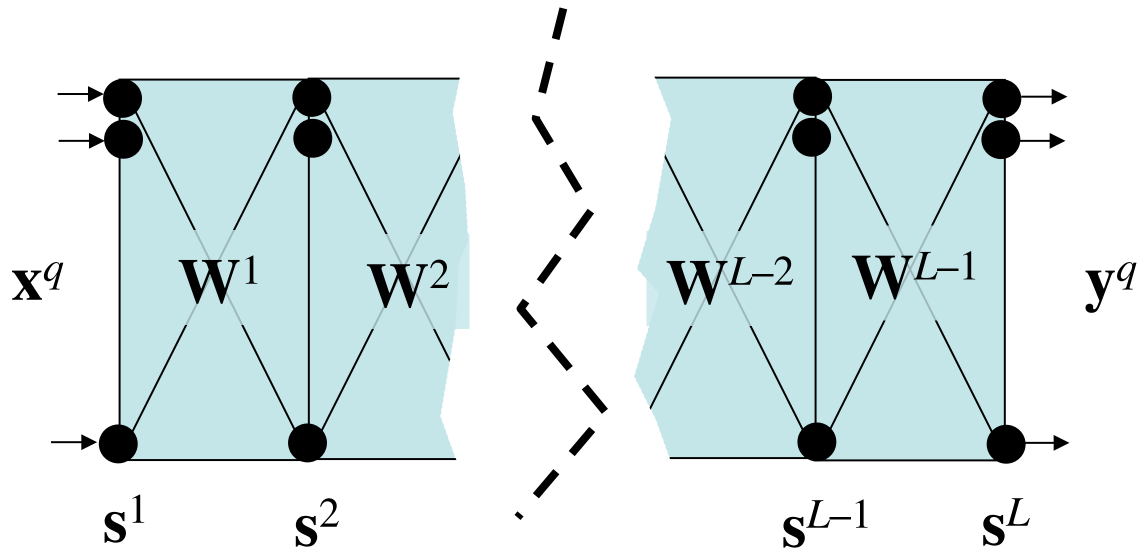
To know how to decrease the differences between actual & desired outputs,

we need to know elements of Jacobian, $\partial y_j^q / \partial P_k$,

which says how *j*th output varies with *k*th parameter (given the *q*th input)

The Jacobian depends on the specific form of the system, in this case, a feedforward neural network

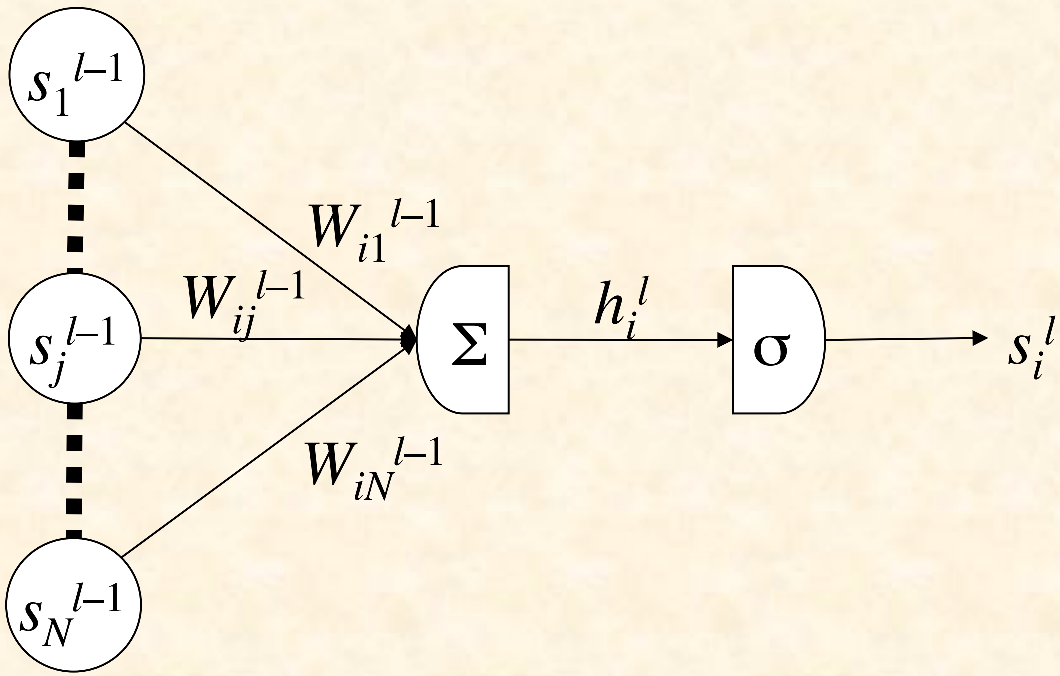
Multilayer Notation



Notation

- L layers of neurons labeled 1, …, L
- N_l neurons in layer l
- $\mathbf{s}^l$ = vector of outputs from neurons in layer l
- input layer $\mathbf{s}^1 = \mathbf{x}^q$ (the input pattern)
- output layer $\mathbf{s}^L = \mathbf{y}^q$ (the actual output)
- $\mathbf{W}^l$ = weights between layers l and l+1
- Problem: find how outputs y_i^q vary with weights W_{jk}^l (l = 1, …, L–1)

Typical Neuron



Error Back-Propagation

We will compute $\frac{\partial E^q}{\partial W_{ij}^l}$ starting with last layer ($l = L - 1$)

and working back to earlier layers ($l = L - 2, \ldots, 1$)

Delta Values

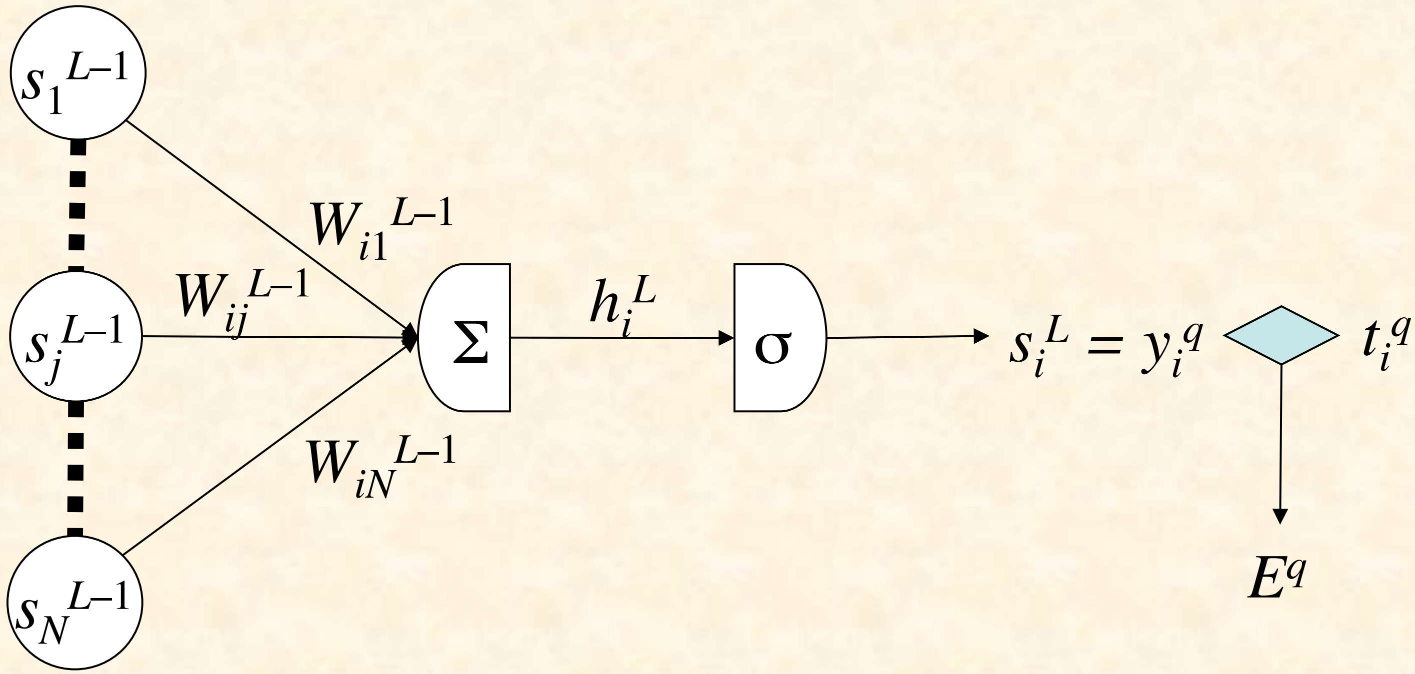
Convenient to break derivatives by chain rule:

$$\frac{\partial E^q}{\partial W_{ij}^{l-1}} = \frac{\partial E^q}{\partial h_i^l} \frac{\partial h_i^l}{\partial W_{ij}^{l-1}}$$

Let $\delta_i^l = \dfrac{\partial E^q}{\partial h_i^l}$

So $\dfrac{\partial E^q}{\partial W_{ij}^{l-1}} = \delta_i^l \dfrac{\partial h_i^l}{\partial W_{ij}^{l-1}}$

Output-Layer Neuron



Output-Layer Derivatives (1)

$$\delta_i^L = \frac{\partial E^q}{\partial h_i^L} = \frac{\partial}{\partial h_i^L} \sum_k \left(s_k^L - t_k^q\right)^2$$

$$= \frac{\mathrm{d}\left(s_i^L - t_i^q\right)^2}{\mathrm{d}\, h_i^L} = 2\left(s_i^L - t_i^q\right)\frac{\mathrm{d}\, s_i^L}{\mathrm{d}\, h_i^L}$$

$$= 2\left(s_i^L - t_i^q\right)\sigma'\left(h_i^L\right)$$

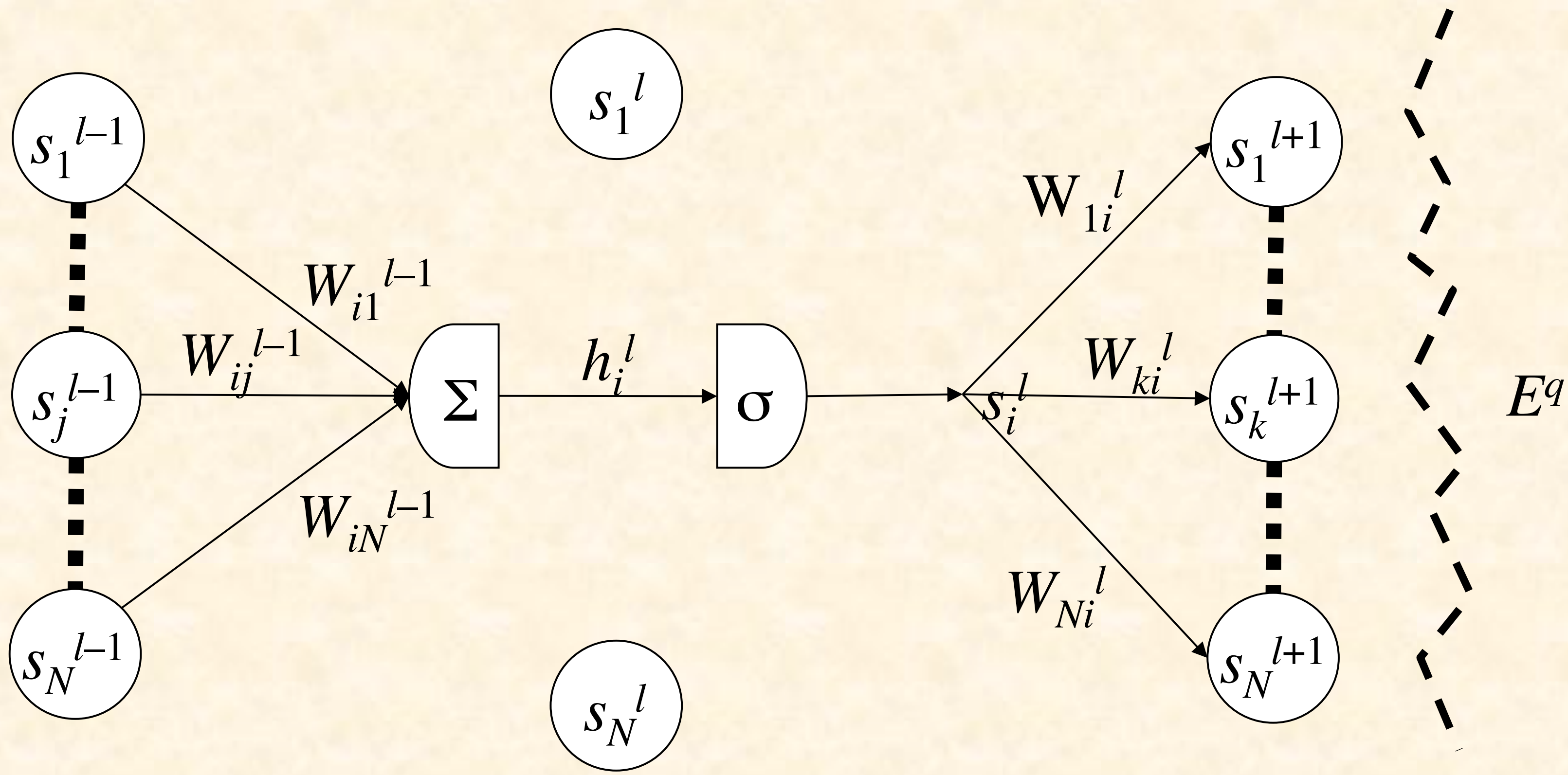
Output-Layer Derivatives (2)

$$\frac{\partial h_i^L}{\partial W_{ij}^{L-1}} = \frac{\partial}{\partial W_{ij}^{L-1}} \sum_k W_{ik}^{L-1} s_k^{L-1} = s_j^{L-1}$$

$$\therefore \frac{\partial E^q}{\partial W_{ij}^{L-1}} = \delta_i^L s_j^{L-1}$$

$$\text{where } \delta_i^L = 2\left(s_i^L - t_i^q\right)\sigma'\left(h_i^L\right)$$

Hidden-Layer Neuron



Hidden-Layer Derivatives (1)

Recall $\frac{\partial E^q}{\partial W_{ij}^{l-1}} = \delta_i^l \frac{\partial h_i^l}{\partial W_{ij}^{l-1}}$

$$\delta_i^l = \frac{\partial E^q}{\partial h_i^l} = \sum_k \frac{\partial E^q}{\partial h_k^{l+1}} \frac{\partial h_k^{l+1}}{\partial h_i^l} = \sum_k \delta_k^{l+1} \frac{\partial h_k^{l+1}}{\partial h_i^l}$$

$$\frac{\partial h_k^{l+1}}{\partial h_i^l} = \frac{\partial \sum_m W_{km}^l s_m^l}{\partial h_i^l} = \frac{\partial W_{ki}^l s_i^l}{\partial h_i^l} = W_{ki}^l \frac{\mathrm{d}\sigma\left(h_i^l\right)}{\mathrm{d}\,h_i^l} = W_{ki}^l \sigma'\left(h_i^l\right)$$

$$\therefore \delta_i^l = \sum_k \delta_k^{l+1} W_{ki}^l \sigma'\left(h_i^l\right) = \sigma'\left(h_i^l\right) \sum_k \delta_k^{l+1} W_{ki}^l$$

Hidden-Layer Derivatives (2)

$$\frac{\partial h_i^l}{\partial W_{ij}^{l-1}} = \frac{\partial}{\partial W_{ij}^{l-1}} \sum_k W_{ik}^{l-1} s_k^{l-1} = \frac{\mathrm{d} W_{ij}^{l-1} s_j^{l-1}}{\mathrm{d} W_{ij}^{l-1}} = s_j^{l-1}$$

$$\therefore \frac{\partial E^q}{\partial W_{ij}^{l-1}} = \delta_i^l s_j^{l-1}$$

$$\text{where } \delta_i^l = \sigma'\left(h_i^l\right) \sum_k \delta_k^{l+1} W_{ki}^l$$

Derivative of Sigmoid

Suppose $s = \sigma(h) = \dfrac{1}{1 + \exp(-\alpha h)}$ (logistic sigmoid)

$$\mathrm{D}_h\, s = \mathrm{D}_h\left[1 + \exp(-\alpha h)\right]^{-1} = -\left[1 + \exp(-\alpha h)\right]^{-2} \mathrm{D}_h\left(1 + e^{-\alpha h}\right)$$

$$= -\left(1 + e^{-\alpha h}\right)^{-2}\left(-\alpha e^{-\alpha h}\right) = \alpha \frac{e^{-\alpha h}}{\left(1 + e^{-\alpha h}\right)^2}$$

$$= \alpha \frac{1}{1 + e^{-\alpha h}} \frac{e^{-\alpha h}}{1 + e^{-\alpha h}} = \alpha s\left(\frac{1 + e^{-\alpha h}}{1 + e^{-\alpha h}} - \frac{1}{1 + e^{-\alpha h}}\right)$$

$$= \alpha s(1 - s)$$

Summary of Back-Propagation Algorithm

Output layer : $\delta_i^L = 2\alpha s_i^L \left(1 - s_i^L\right)\left(s_i^L - t_i^q\right)$

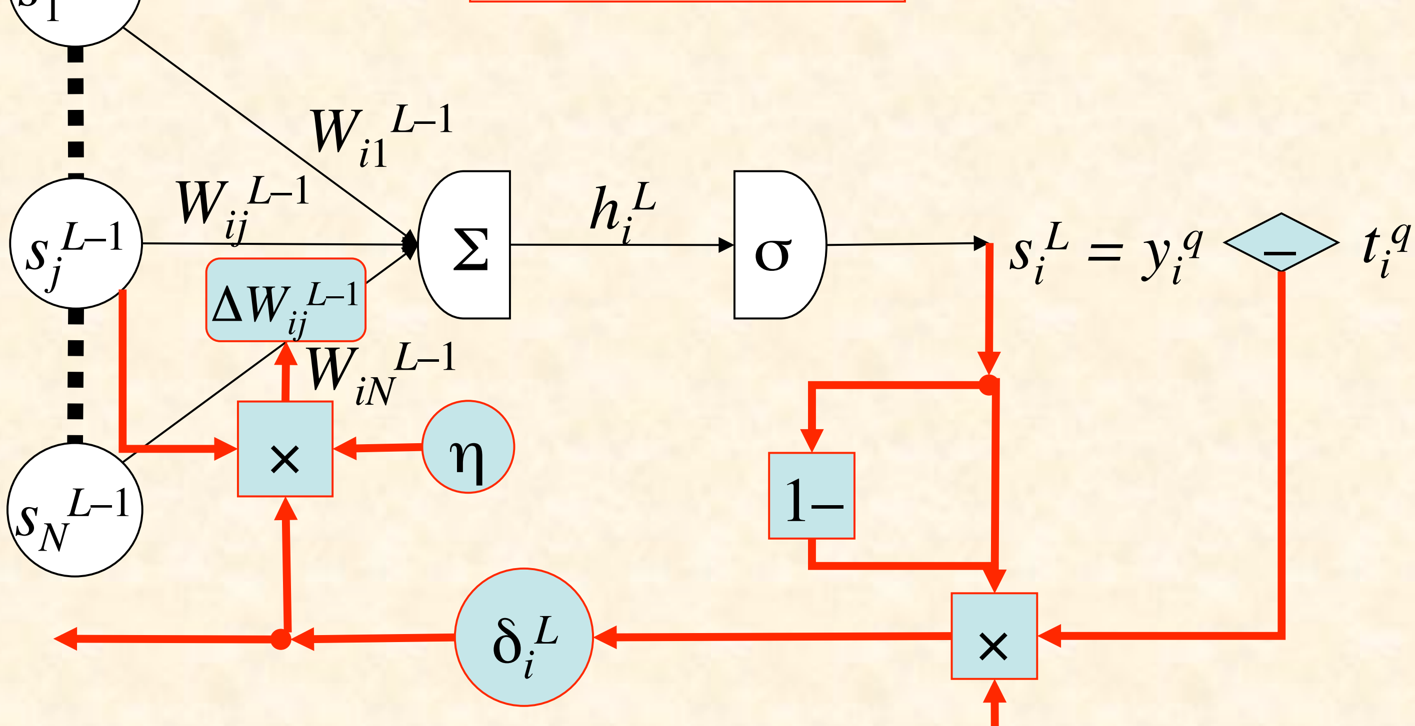
$$\frac{\partial E^q}{\partial W_{ij}^{L-1}} = \delta_i^L s_j^{L-1}$$

Hidden layers : $\delta_i^l = \alpha s_i^l \left(1 - s_i^l\right) \sum_k \delta_k^{l+1} W_{ki}^l$

$$\frac{\partial E^q}{\partial W_{ij}^{l-1}} = \delta_i^l s_j^{l-1}$$

Output-Layer Computation

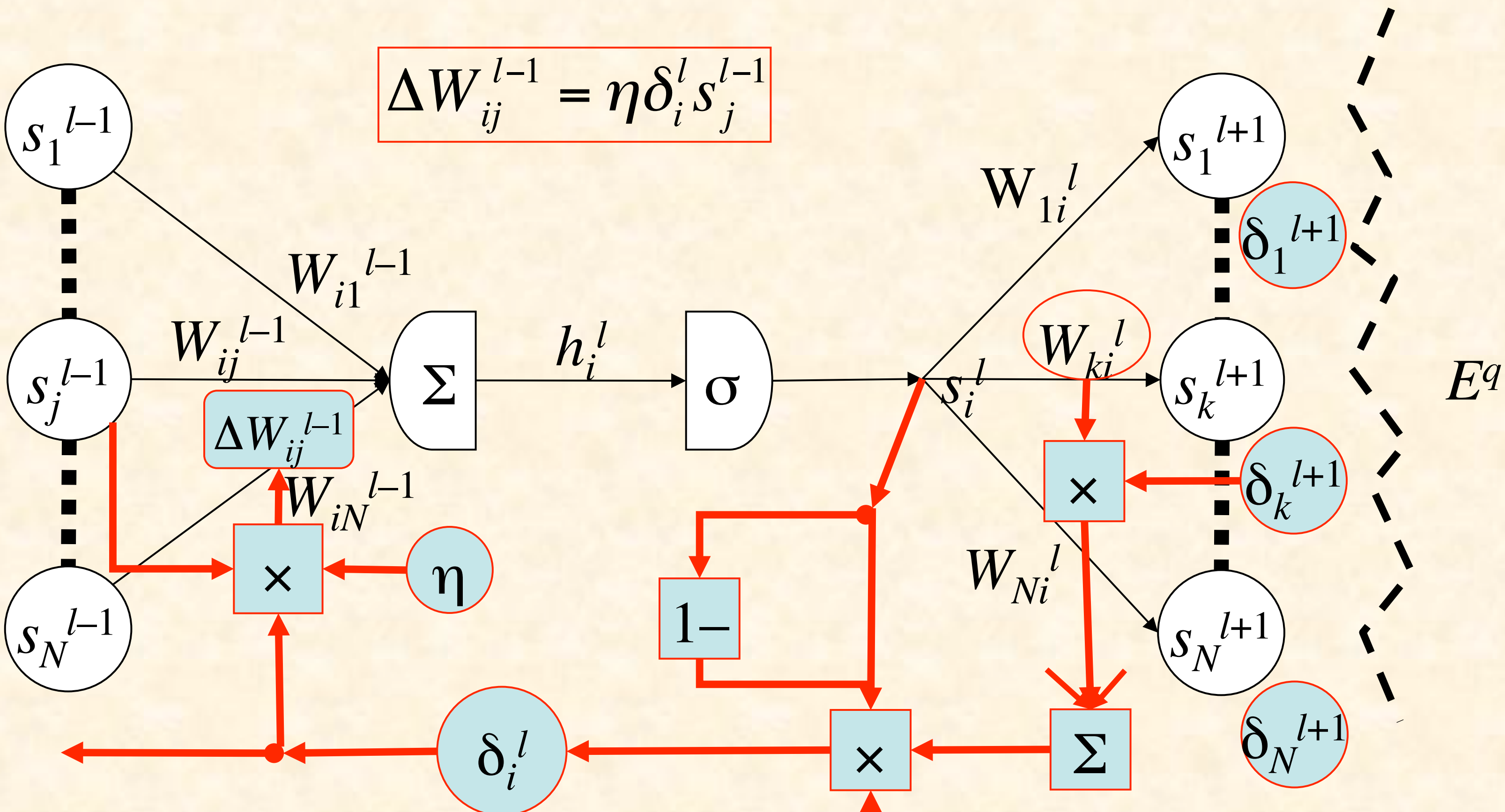
$$\Delta W_{ij}^{L-1} = \eta \delta_i^L s_j^{L-1}$$



$$\delta_i^L = 2\alpha s_i^L \left(1 - s_i^L\right)\left(t_i^q - s_i^L\right)$$

Hidden-Layer Computation

$$\Delta W_{ij}^{l-1} = \eta \delta_i^l s_j^{l-1}$$

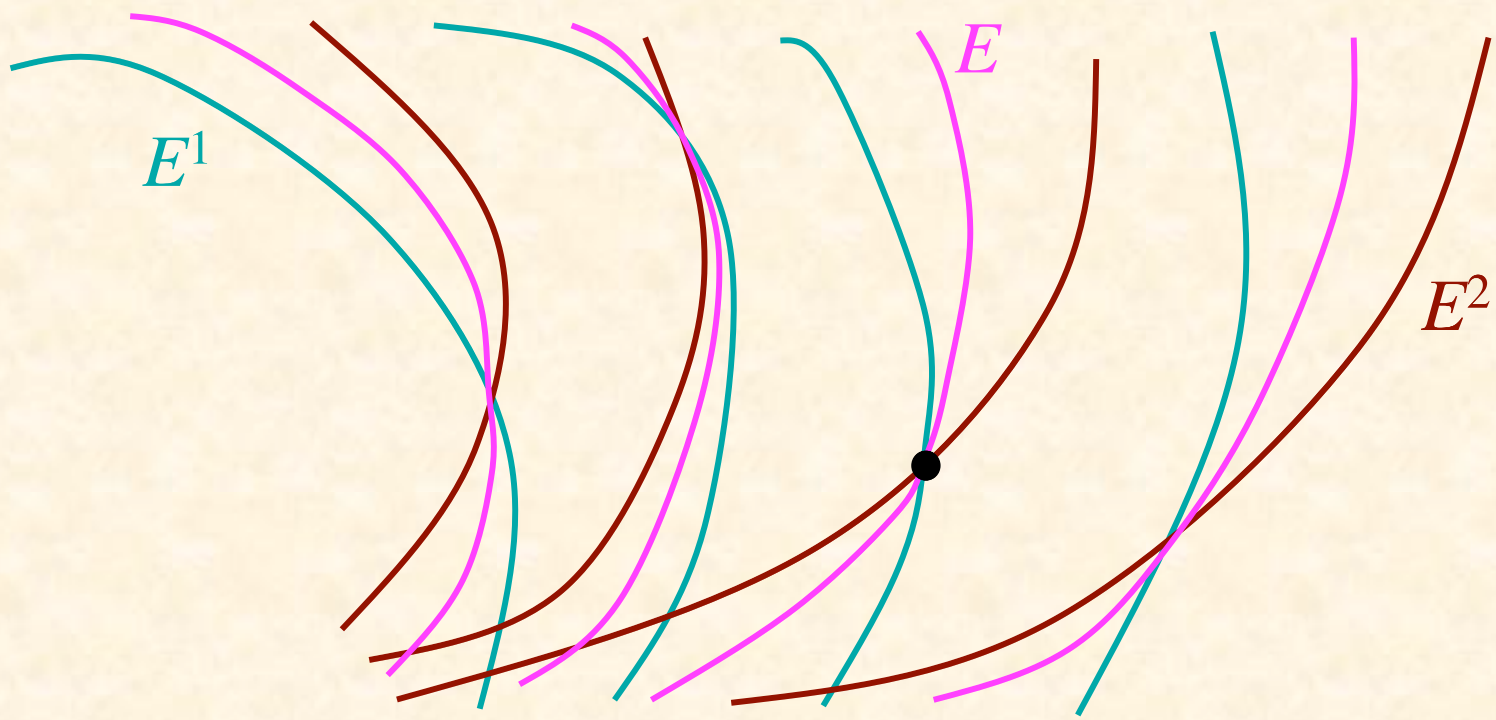


$$\delta_i^l = \alpha s_i^l \left(1 - s_i^l\right) \sum_k \delta_k^{l+1} W_{ki}^l$$

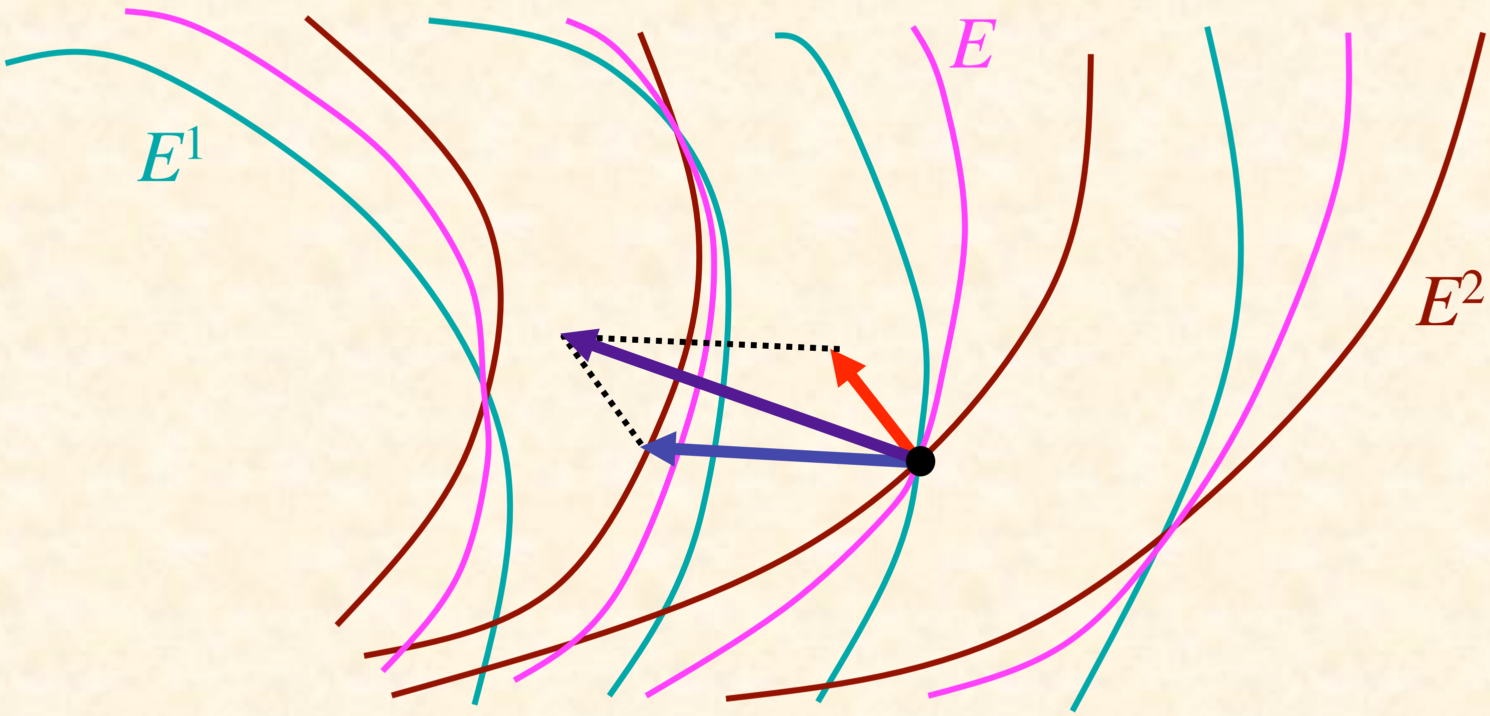
Training Procedures

- Batch Learning
  - on each *epoch* (pass through all the training pairs),
  - weight changes for all patterns accumulated
  - weight matrices updated at end of epoch
  - accurate computation of gradient
- Online Learning
  - weight are updated after back-prop of each training pair
  - usually randomize order for each epoch
  - approximation of gradient
- Doesn't make much difference

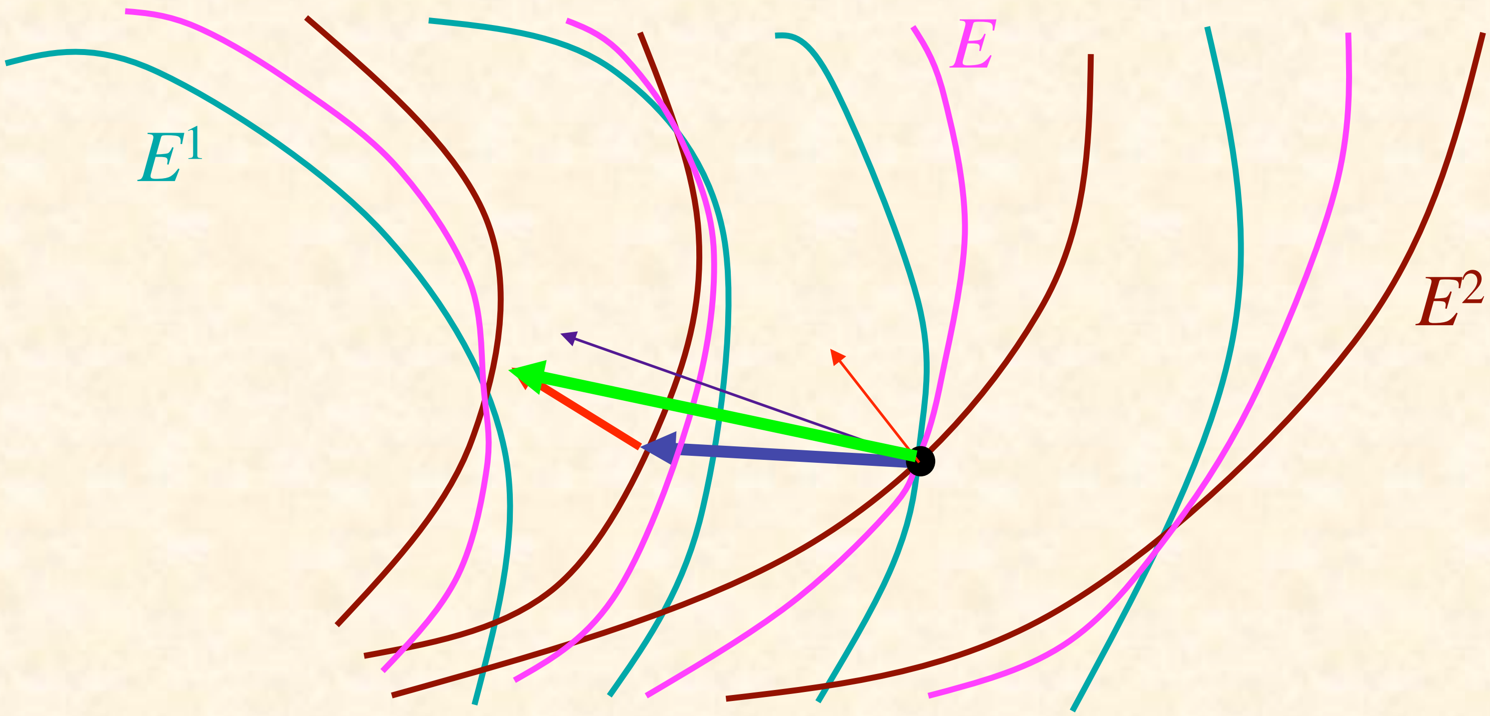
Summation of Error Surfaces



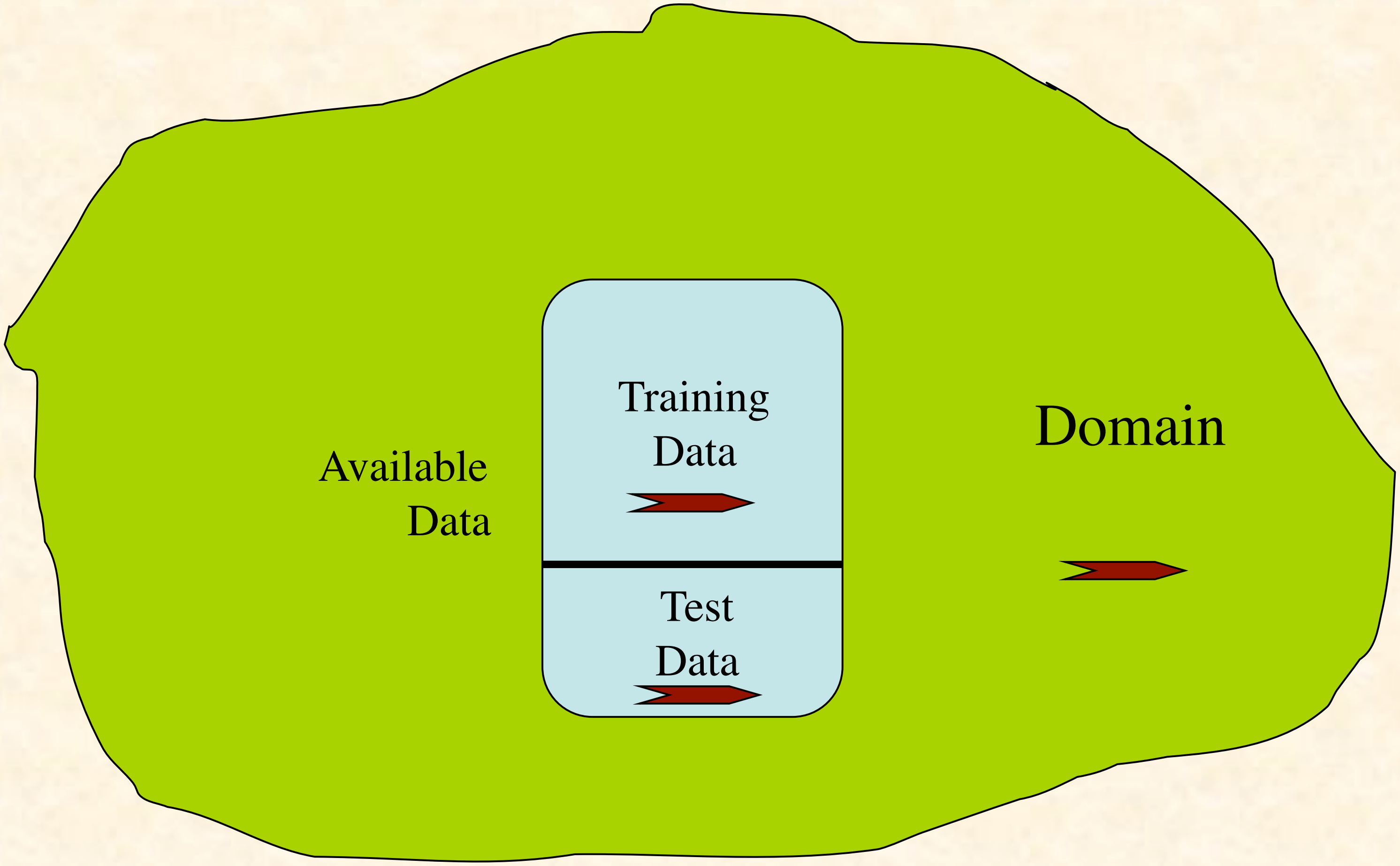
Gradient Computation in Batch Learning



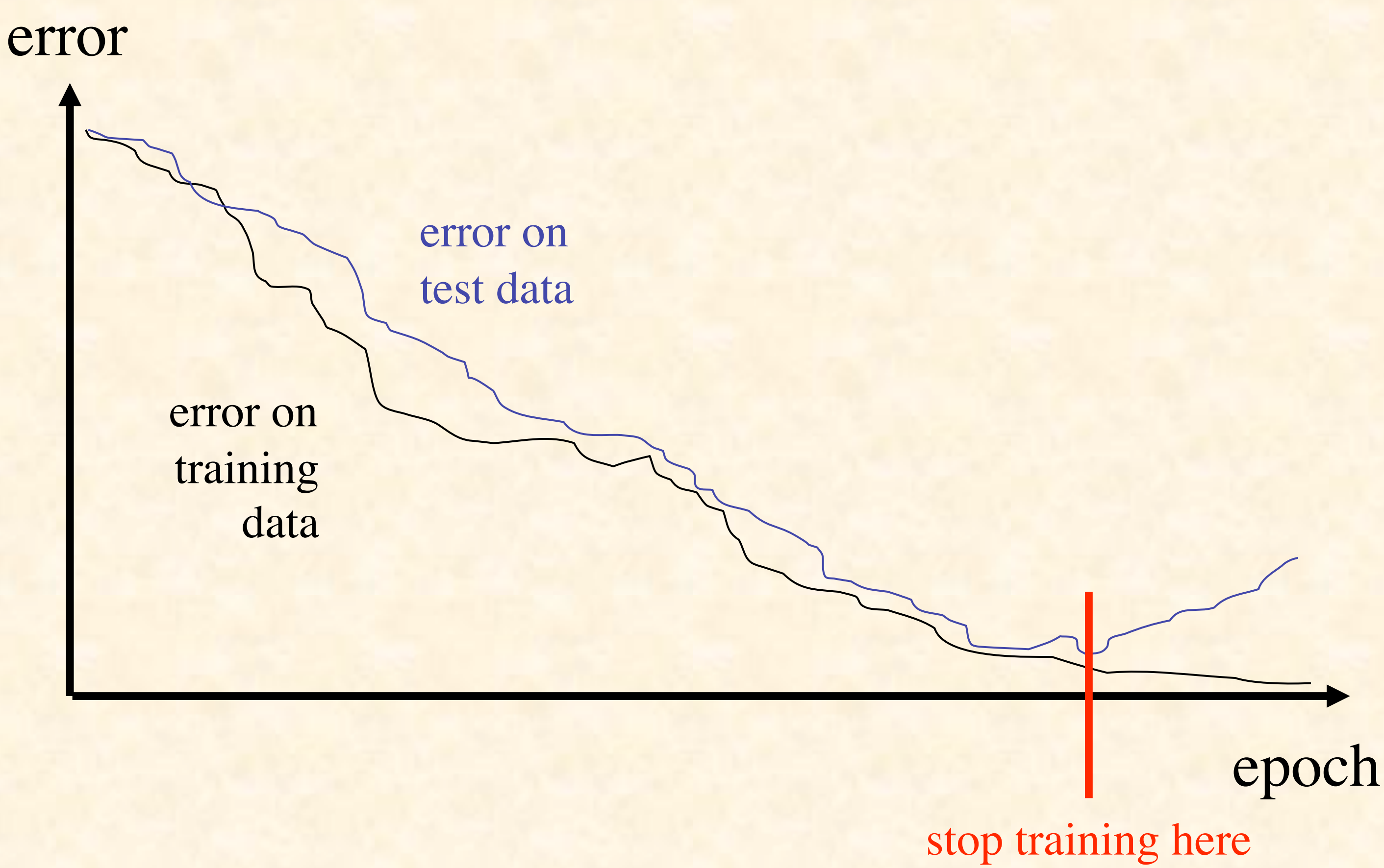
Gradient Computation in Online Learning



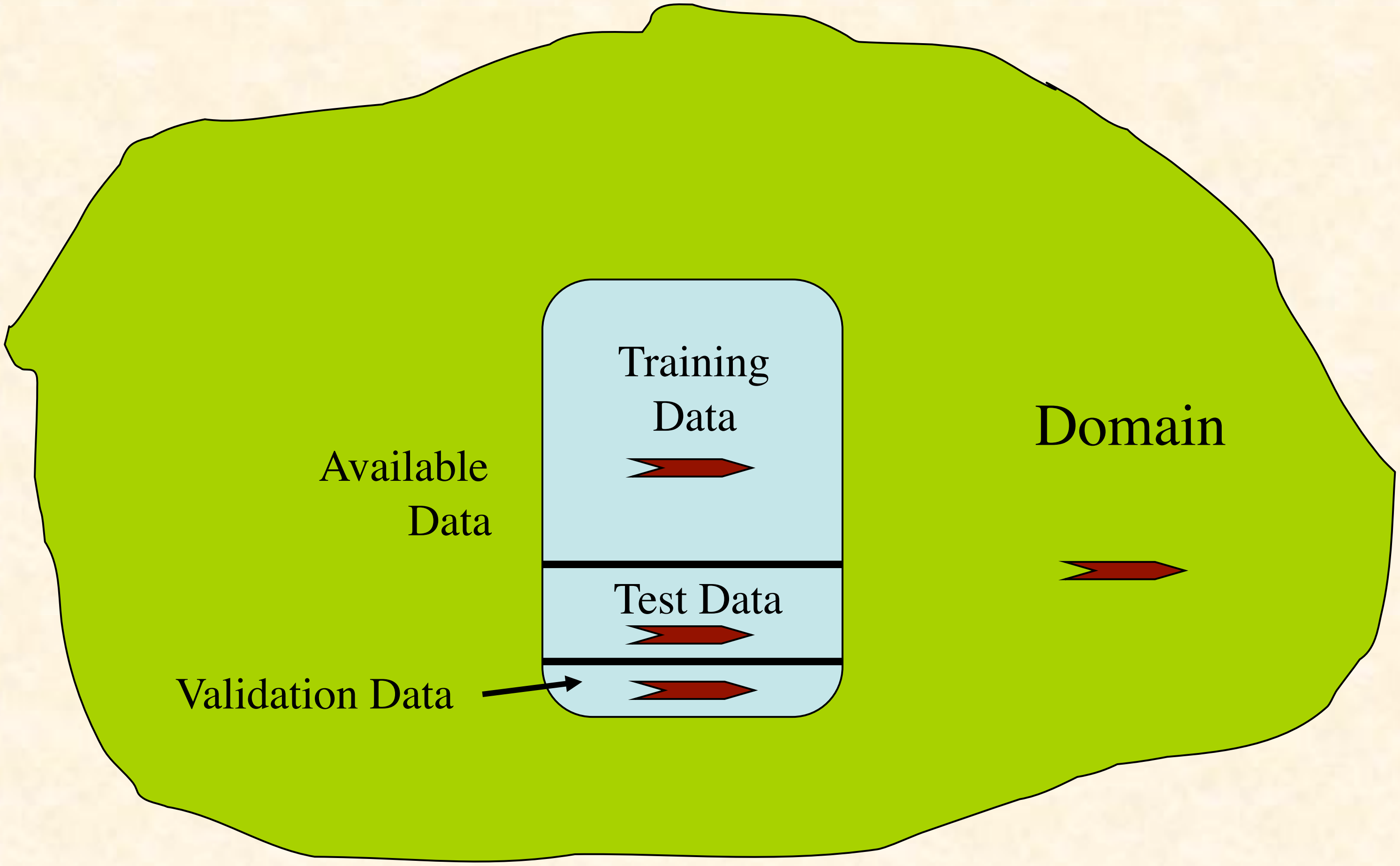
Testing Generalization



Problem of Rote Learning



Improving Generalization



A Few Random Tips

- Too few neurons and the ANN may not be able to decrease the error enough
- Too many neurons can lead to rote learning
- Preprocess data to:
  - standardize
  - eliminate irrelevant information
  - capture invariances
  - keep relevant information
- If stuck in local min., restart with different random weights

Run Example BP Learning

Beyond Back-Propagation

- Adaptive Learning Rate
- Adaptive Architecture
  - Add/delete hidden neurons
  - Add/delete hidden layers
- Radial Basis Function Networks
- Recurrent BP
- Etc., etc., etc.…

What is the Power of Artificial Neural Networks?

- With respect to Turing machines?
- As function approximators?

Can ANNs Exceed the "Turing Limit"?

- There are many results, which depend sensitively on assumptions; for example:
- Finite NNs with real-valued weights have super-Turing power (Siegelmann & Sontag '94)
- Recurrent nets with Gaussian noise have sub-Turing power (Maass & Sontag '99)
- Finite recurrent nets with real weights can recognize <u>all</u> languages, and thus are super-Turing (Siegelmann '99)
- Stochastic nets with rational weights have super-Turing power (but only P/POLY, BPP/log*) (Siegelmann '99)
- But computing classes of functions is not a very relevant way to evaluate the capabilities of neural computation

A Universal Approximation Theorem

Suppose f is a continuous function on $[0,1]^n$

Suppose σ is a nonconstant, bounded,

monotone increasing real function on $\Re$.

For any $\varepsilon > 0$, there is an m such that

$\exists \mathbf{a} \in \Re^m$, $\mathbf{b} \in \Re^n$, $\mathbf{W} \in \Re^{m \times n}$ such that if

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{m} a_i \sigma\left(\sum_{j=1}^{n} W_{ij} x_j + b_j\right)$$

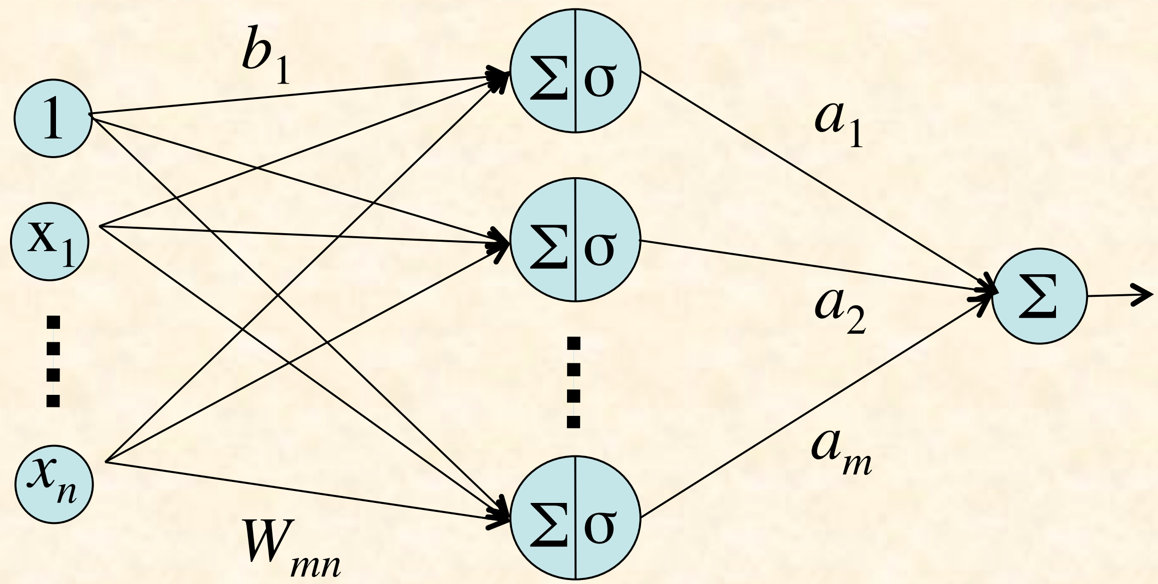
$$\left[\text{i.e., } F(\mathbf{x}) = \mathbf{a} \cdot \sigma(\mathbf{W}\mathbf{x} + \mathbf{b})\right]$$

then $\left|F(\mathbf{x}) - f(\mathbf{x})\right| < \varepsilon$ for all $\mathbf{x} \in [0,1]^n$

(see, e.g., Haykin, *N.Nets* 2/e, 208–9)

One Hidden Layer is Sufficient

- Conclusion: One hidden layer is sufficient to approximate any continuous function arbitrarily closely



The Golden Rule of Neural Nets

Neural Networks are the

second-best way

to do *everything*!