1. (2.3.4) There are 11 warehouses, 5 of which contain a particular type of item. If warehouses are called in random order, what is the probability that the first call to a warehouse having the item is the \(i\)th call? What is the expected number of calls, assuming calling stops when a warehouse having the item is contacted?

Let \(f\) be the random variable representing the number of calls up to and including the first to a warehouse containing the item. We are being asked for the expectation of \(f\),

\[
\mathcal{E} f = \sum_z z P_f(z) \tag{1}
\]

According to the homework 3 handout, \(f\) has probability mass function

\[
P_f(i) = [i \in \mathbb{Z}^+] P(C_i)
\]

\[
= [i \in \mathbb{Z}^+] P(C_i | \bigcap_k C_k) \prod_{j<i} P(C'_j | \bigcap_k C'_k) 
\]

\[
= [i \in \mathbb{Z}^+] \frac{5}{11 - (i - 1)} [i - 1 \leq 6] \prod_{j<i} \left(1 - \frac{5}{11 - (j - 1)} [j - 1 \leq 6] \right)
\]

Therefore,

<table>
<thead>
<tr>
<th>(i)</th>
<th>(P_f(i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5/11</td>
</tr>
<tr>
<td>2</td>
<td>3/11</td>
</tr>
<tr>
<td>3</td>
<td>5/33</td>
</tr>
<tr>
<td>4</td>
<td>5/66</td>
</tr>
<tr>
<td>5</td>
<td>5/154</td>
</tr>
<tr>
<td>6</td>
<td>5/462</td>
</tr>
<tr>
<td>7</td>
<td>1/462</td>
</tr>
<tr>
<td>otherwise</td>
<td>0</td>
</tr>
</tbody>
</table>

It follows from (1) that \(\mathcal{E} f = 2\).
2. (2.3.12) Car panels are machine spray-painted; the random variable \( t \) measuring the paint thickness in millimeters has density

\[
\rho_t(x) = [0.12 \leq x \leq 0.6] \frac{15625(0.6 - (x - 0.2)^2)}{4164}
\]

What is the expected thickness? What is the median thickness?

The expectation is

\[
E(t) = \int x \rho_t(x) \, dx
\]

\[
= \int_{0.12}^{0.6} x \frac{15625(0.6 - (x - 0.2)^2)}{4164} \, dx
\]

\[
\approx 0.3489372
\]

The median thickness is the solution to \( F_t(x) = 0.5 \), thus we want

\[
0.5 = \int_{-\infty}^{x} \rho_t(z) \, dz = \int_{0.12}^{x} \frac{15625(0.6 - (z - 0.2)^2)}{4164} \, dz
\]

Solving the above gives \( \approx 0.344 \)

3. (2.3.16) A consultant has 11 open appointment times; 4 on Wednesday and 7 on Friday. Suppose clients make appointments by randomly choosing a remaining open time. Let the random variable \( f \) be the number of appointments up to and including when Friday’s schedule has just been filled. What is the expected value of \( f \)?

According to the homework 3 handout, \( f \) has probability mass function

\[
P_f(x) = [7 \leq x \leq 11] \frac{(x-1)}{11 \choose 7}
\]

It follows that

\[
E_f = \sum_z z P_f(z) = \sum_z z \frac{(z-1)}{11 \choose 7} = 10.5
\]
4. (2.4.7) Car panels are machine spray-painted; the random variable $t$ measuring the paint thickness in millimeters has density

$$
\rho_t(x) = \begin{cases} 
15625(0.6 - (x - 0.2)^2) \\
0 & \text{if } 0.12 \leq x \leq 0.6 \\
4164 & \text{otherwise}
\end{cases}
$$

What is the standard deviation of paint thickness? What is the interquartile range?

The variance is

$$
\mathcal{E}(t^2) - \mathcal{E}(t)^2 \approx \int x^2 \rho_t(x) \, dx - (0.34893372)^2
$$

$$
= \int_{0.12}^{0.6} x^2 \frac{15625(0.6 - (x - 0.2)^2)}{4164} \, dx - (0.34893372)^2
$$

$$
\approx 0.0185464
$$

The standard deviation is therefore

$$
\sigma \approx \sqrt{0.0185464} \approx 0.1362
$$

The upper quartile is the solution to $\mathcal{F}_t(x) = 0.75$, thus we want

$$
0.75 = \int_{-\infty}^{x} \rho_t(z) \, dz = \int_{0.12}^{x} \frac{15625(0.6 - (z - 0.2)^2)}{4164} \, dz
$$

Solving the above gives $\approx 0.4635775$. The lower quartile is the solution to $\mathcal{F}_t(x) = 0.25$, thus we want

$$
0.25 = \int_{-\infty}^{x} \rho_t(z) \, dz = \int_{0.12}^{x} \frac{15625(0.6 - (z - 0.2)^2)}{4164} \, dz
$$

Solving the above gives $\approx 0.2313415$. The interquartile range is therefore

$$
\approx 0.4635775 - 0.2313415 = 0.232236
$$
5. (2.4.10) The time $t$ taken to serve a customer at a fast-food restaurant has a mean of 80 seconds and a standard deviation of 10 seconds. Use Chebyshev’s inequality to calculate time intervals that have (at least) 70% and 90% probabilities of containing a particular service time.

The mean is $\mu = 80$ and the standard deviation is $\sigma = 10$. Chebyshev’s inequality implies

$$P(\mu - c\sigma \leq t \leq \mu + c\sigma) \geq 1 - \frac{1}{c^2} \quad (2)$$

The right hand side of (2) is 0.70 for $c \approx 1.82$, and the corresponding time interval is

$$[\mu - c\sigma, \mu + c\sigma] = [61.7, 98.3]$$

The right hand side of (2) is 0.90 for $c \approx 3.16$, and the corresponding time interval is

$$[\mu - c\sigma, \mu + c\sigma] = [48.3, 111.6]$$

6. (2.4.12) A consultant has 11 open appointment times; 4 on Wednesday and 7 on Friday. Suppose clients make appointments by randomly choosing a remaining open time. Let the random variable $f$ be the number of appointments up to and including when Friday’s schedule has just been filled. What is the standard deviation of $f$?

According to the homework 3 handout, $f$ has probability mass function

$$P_f(x) = \begin{cases} 7 \leq x \leq 11 & \frac{(x-1)}{\binom{11}{7}} \\ \end{cases}$$

It follows that

$$\text{var}(f) = \sum_z z^2 P_f(z) - (\mathbb{E} f)^2$$

$$= \sum_{z=7}^{11} z^2 \frac{(z-1)}{\binom{11}{7}} - (10.5)^2$$

$$\approx 0.58333$$

Hence the standard deviation is $\approx 0.76376$. 