9.7. Summary of key points

1. Negative feedback causes the system output to closely follow the reference input, according to the gain $1/H(s)$. The influence on the output of disturbances and variation of gains in the forward path is reduced.

2. The loop gain $\gamma(s)$ is equal to the products of the gains in the forward and feedback paths. The loop gain is a measure of how well the feedback system works: a large loop gain leads to better regulation of the output. The crossover frequency $f_c$ is the frequency at which the loop gain $\gamma$ has unity magnitude, and is a measure of the bandwidth of the control system.
Summary of key points

3. The introduction of feedback causes the transfer functions from disturbances to the output to be multiplied by the factor \( \frac{1}{1 + T(s)} \). At frequencies where \( T \) is large in magnitude (i.e., below the crossover frequency), this factor is approximately equal to \( \frac{1}{T(s)} \). Hence, the influence of low-frequency disturbances on the output is reduced by a factor of \( \frac{1}{T(s)} \). At frequencies where \( T \) is small in magnitude (i.e., above the crossover frequency), the factor is approximately equal to 1. The feedback loop then has no effect. Closed-loop disturbance-to-output transfer functions, such as the line-to-output transfer function or the output impedance, can easily be constructed using the algebra-on-the-graph method.

4. Stability can be assessed using the phase margin test. The phase of \( T \) is evaluated at the crossover frequency, and the stability of the important closed-loop quantities \( T(1+T) \) and \( 1/(1+T) \) is then deduced. Inadequate phase margin leads to ringing and overshoot in the system transient response, and peaking in the closed-loop transfer functions.

Summary of key points

5. Compensators are added in the forward paths of feedback loops to shape the loop gain, such that desired performance is obtained. Lead compensators, or \( PD \) controllers, are added to improve the phase margin and extend the control system bandwidth. \( PI \) controllers are used to increase the low-frequency loop gain, to improve the rejection of low-frequency disturbances and reduce the steady-state error.

6. Loop gains can be experimentally measured by use of voltage or current injection. This approach avoids the problem of establishing the correct quiescent operating conditions in the system, a common difficulty in systems having a large dc loop gain. An injection point must be found where interstage loading is not significant. Unstable loop gains can also be measured.
Part III. Magnetics

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14 Inductor Design
15 Transformer Design

Chapter 13 Basic Magnetics Theory

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13.5 Several Types of Magnetic Devices, Their $B$–$H$ Loops, and Core vs. Copper Loss
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13.6 Summary of Key Points

13.1 Review of Basic Magnetics
   13.1.1 Basic relationships

- Faraday’s law
  - $v(t)$
  - $B(t)$, $\Phi(t)$

- Terminal characteristics
  - $i(t)$

- Core characteristics
  - $H(t)$, $\mathcal{H}(t)$

- Ampere’s law
Magnetic field $H$ and magnetomotive force $\mathcal{F}$

Magnetomotive force (MMF) $\mathcal{F}$ between points $x_1$ and $x_2$ is related to the magnetic field $H$ according to

$$\mathcal{F} = \int_{x_1}^{x_2} H \cdot dt$$

**Example:** uniform magnetic field of magnitude $H$

Length $l$

Magnetic field $H$

MMF $\mathcal{F} = HL$

**Analogous to electric field of strength $E$, which induces voltage (EMF) $V$:**

Length $l$

Electric field $E$

Voltage $V = EL$
**Flux density $B$ and total flux $\Phi$**

The total magnetic flux $\Phi$ passing through a surface of area $A_c$ is related to the flux density $B$ according to

$$\Phi = \int_{\text{surface } S} B \cdot dA$$

**Example: uniform flux density of magnitude $B$**

**Analogous to electrical conductor current density of magnitude $J$, which leads to total conductor current $I$.**

**Faraday’s law**

Voltage $v(t)$ is induced in a loop of wire by change in the total flux $\Phi(t)$ passing through the interior of the loop, according to

$$v(t) = \frac{d\Phi(t)}{dt}$$

For uniform flux distribution, $\Phi(t) = B(t)A_c$, and hence

$$v(t) = A_c \frac{dB(t)}{dt}$$
**Lenz’s law**

The voltage $v(t)$ induced by the changing flux $\Phi(t)$ is of the polarity that tends to drive a current through the loop to counteract the flux change.

**Example: a shorted loop of wire**

- Changing flux $\Phi(t)$ induces a voltage $v(t)$ around the loop
- This voltage, divided by the impedance of the loop conductor, leads to current $i(t)$
- This current induces a flux $\Phi'(t)$, which tends to oppose changes in $\Phi(t)$

**Ampere’s law**

The net MMF around a closed path is equal to the total current passing through the interior of the path:

$$\oint_{\text{closed path}} H \cdot dl = \text{total current passing through interior of path}$$

**Example: magnetic core. Wire carrying current $i(t)$ passes through core window.**

- Illustrated path follows magnetic flux lines around interior of core
- For uniform magnetic field strength $H(t)$, the integral (MMF) is $H(t)L_m$. So
  $$\mathcal{F}(t) = H(t)L_m = i(t)$$
Ampere’s law: discussion

- Relates magnetic field strength $H(t)$ to winding current $i(t)$
- We can view winding currents as sources of MMF
- Previous example: total MMF around core, $\mathcal{F}(t) = H(t)L_A$, is equal to the winding current MMF $i(t)$
- The total MMF around a closed loop, accounting for winding current MMF’s, is zero

Core material characteristics: the relation between $B$ and $H$

$\mu_0 = \text{permeability of free space}$
$= 4\pi \cdot 10^{-7} \text{Henries per meter}$

$B = \mu_0 H$

$A \text{ magnetic core material}$

Highly nonlinear, with hysteresis and saturation
Piecewise-linear modeling of core material characteristics

**No hysteresis or saturation**

\[ B = \mu H \]
\[ \mu = \mu_r \mu_0 \]

Typical \( \mu_r = 10^3 \) to \( 10^5 \)

**Saturation, no hysteresis**

\[ B = B_{sat} \]

Typical \( B_{sat} = 0.3 \) to \( 0.5T \), ferrite

0.5 to \( 1T \), powdered iron

1 to \( 2T \), iron laminations

---

Units

Table 12.1. Units for magnetic quantities

<table>
<thead>
<tr>
<th>Quantity</th>
<th>MKS</th>
<th>SiSI (uninitialized)</th>
<th>Conversions</th>
</tr>
</thead>
<tbody>
<tr>
<td>core material equation</td>
<td>( B = \mu_r \mu_0 H )</td>
<td>( B = \mu_i H )</td>
<td></td>
</tr>
<tr>
<td>( B )</td>
<td>Tesla</td>
<td>Gauss</td>
<td>( 1T = 10^4 )G</td>
</tr>
<tr>
<td>( H )</td>
<td>Ampere / meter</td>
<td>Oersted</td>
<td>( 1A/m = 4\pi \cdot 10^{-7} ) Oe</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>Weber</td>
<td>Maxwell</td>
<td>( 1Wb = 10^8 ) Mx</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( 1T = 1Wb / m^2 )</td>
</tr>
</tbody>
</table>
Example: a simple inductor

Faraday’s law:
For each turn of wire, we can write
\[ v_{\text{ind}}(t) = \frac{d\Phi(t)}{dt} \]

Total winding voltage is
\[ v(t) = n v_{\text{ind}}(t) = n \frac{d\Phi(t)}{dt} \]

Express in terms of the average flux density \( B(t) = \mathcal{A}(t)/A_c \)
\[ v(t) = n A_c \frac{dB(t)}{dt} \]

Inductor example: Ampere’s law

Choose a closed path which follows the average magnetic field line around the interior of the core. Length of this path is called the mean magnetic path length \( \ell_m \).

For uniform field strength \( H(t) \), the core MMF around the path is \( H \ell_m \).

Winding contains \( n \) turns of wire, each carrying current \( i(t) \). The net current passing through the path interior (i.e., through the core window) is \( ni(t) \).

From Ampere’s law, we have
\[ H(t) \ell_m = ni(t) \]
Inductor example: core material model

\[
B = \begin{cases} 
B_{\text{sat}} & \text{for } H \geq B_{\text{sat}}/\mu \\
\mu H & \text{for } |H| < B_{\text{sat}}/\mu \\
-B_{\text{sat}} & \text{for } H \leq -B_{\text{sat}}/\mu 
\end{cases}
\]

Find winding current at onset of saturation: substitute \( i = I_{\text{sat}} \) and \( H = B_{\text{sat}}/\mu \) into equation previously derived via Ampere’s law. Result is:

\[
I_{\text{sat}} = \frac{B_{\text{sat}} L_m}{\mu n}
\]

Electrical terminal characteristics

We have:

\[
v(t) = \frac{n A_c}{L} \frac{dB(t)}{dt} \\
v(t) = \frac{n A_c}{L} \mu H(t) = n H(t)
\]

Eliminate \( B \) and \( H \), and solve for relation between \( v \) and \( i \). For \( |i| < I_{\text{sat}} \):

\[
v(t) = \frac{\mu n A_c}{L} \frac{dH(t)}{dt} \\
v(t) = \frac{\mu n A_c}{L} \frac{di(t)}{dt}
\]

which is of the form

\[
v(t) = L \frac{di(t)}{dt} \\
v(t) = L \frac{di(t)}{dt} \quad \text{with} \quad L = \frac{\mu n^2 A_c}{L_m}
\]

— an inductor

For \( |i| > I_{\text{sat}} \), the flux density is constant and equal to \( B_{\text{sat}} \). Faraday’s law then predicts

\[
v(t) = n A_c \frac{dB_{\text{sat}}}{dt} = 0
\]

— saturation leads to short circuit
13.1.2 Magnetic circuits

Uniform flux and magnetic field inside a rectangular element:

\[ \Phi = Hl \]

MMF between ends of element is:

\[ \mathcal{F} = \mathcal{H}l \]

Since \( H = B / \mu \) and \( B = \Phi / A \), we can express \( \mathcal{F} \) as

\[ \mathcal{F} = \Phi \mathcal{R} \quad \text{with} \quad \mathcal{R} = \frac{l}{\mu A} \]

A corresponding model:

\[ \Phi \quad \mathcal{R} \]

\( \mathcal{R} \) = reluctance of element

Magnetic circuits: magnetic structures composed of multiple windings and heterogeneous elements

- Represent each element with reluctance
- Windings are sources of MMF
- MMF → voltage, flux → current
- Solve magnetic circuit using Kirchoff’s laws, etc.
Magnetic analog of Kirchoff’s current law

Physical structure

Divergence of $B = 0$
Flux lines are continuous and cannot end
Total flux entering a node must be zero

Magnetic circuit

Node

$\Phi_1 + \Phi_2 = \Phi_3$

Magnetic analog of Kirchoff’s voltage law

Follows from Ampere’s law:

$$\oint_{closed\ path} H \cdot dl = \text{total current passing through interior of path}$$

Left-hand side: sum of MMF’s across the reluctances around the closed path
Right-hand side: currents in windings are sources of MMF’s. An $n$-turn winding carrying current $i(t)$ is modeled as an MMF (voltage) source, of value $ni(t)$.
Total MMF’s around the closed path add up to zero.
Example: inductor with air gap

\[ \Phi = n \int i(t) \, dt \]
\[ \mathcal{F} + \mathcal{F}_g = ni \]
\[ ni = \Phi \left( R_c + R_g \right) \]
\[ R_c = \frac{\ell_c}{\mu A_c} \]
\[ R_g = \frac{\ell_g}{\mu_0 A_c} \]
Solution of model

Faraday's law:
\[ v(t) = n \frac{d\Phi(t)}{dt} \]

Substitute for \( \Phi \):
\[ v(t) = n^2 \frac{a^2}{\mathcal{R}_c + \mathcal{R}_g} \frac{dl(t)}{dt} \]

Hence inductance is
\[ L = n^2 \frac{a^2}{\mathcal{R}_c + \mathcal{R}_g} \]

Effect of air gap

\[ ni = \Phi \left( \mathcal{R}_c + \mathcal{R}_g \right) \]

\[ L = \frac{n^2}{\mathcal{R}_c + \mathcal{R}_g} \]

\[ \Phi_{sat} = B_{sat} A_c \]

\[ I_{sat} = \frac{B_{sat} A_c}{n} \left( \mathcal{R}_c + \mathcal{R}_g \right) \]

Effect of air gap:
- decrease inductance
- increase saturation current
- inductance is less dependent on core permeability
13.2 Transformer modeling

Two windings, no air gap:

\[ R = \frac{L_m}{\mu A_c} \]
\[ \mathcal{F}_e = n_1 i_1 + n_2 i_2 \]
\[ \Phi \mathcal{N} = n_1 i_1 + n_2 i_2 \]

Magnetic circuit model:

13.2.1 The ideal transformer

In the ideal transformer, the core reluctance \( R \) approaches zero.

MMF \( \mathcal{F}_e = \Phi \mathcal{N} \) also approaches zero. We then obtain

\[ 0 = n_1 i_1 + n_2 i_2 \]

Also, by Faraday's law,

\[ v_1 = n_1 \frac{d\Phi}{dt} \]
\[ v_2 = n_2 \frac{d\Phi}{dt} \]

Eliminate \( \Phi \):

\[ \frac{d\Phi}{dt} = \frac{v_1}{n_1} = \frac{v_2}{n_2} \]

Ideal transformer equations:

\[ \frac{v_1}{n_1} = \frac{v_2}{n_2} \quad \text{and} \quad n_1 i_1 + n_2 i_2 = 0 \]
13.2.2 The magnetizing inductance

For nonzero core reluctance, we obtain
\[ \Phi = n_1 i_1 + n_2 i_2 \quad \text{with} \quad v_1 = n_1 \frac{d\Phi}{dt} \]

Eliminate \( \Phi \):
\[ v_1 = \frac{n_1^2}{\Phi} \frac{d}{dt} \left[ i_1 + \frac{n_2}{n_1} i_2 \right] \]

This equation is of the form
\[ v_1 = L_M \frac{di_M}{dt} \]
with
\[ L_M = \frac{n_1^2}{\Phi} \]
\[ i_M = i_1 + \frac{n_2}{n_1} i_2 \]

Magnetizing inductance: discussion

- Models magnetization of core material
- A real, physical inductor, that exhibits saturation and hysteresis
- If the secondary winding is disconnected:
  - we are left with the primary winding on the core
  - primary winding then behaves as an inductor
  - the resulting inductor is the magnetizing inductance, referred to the primary winding
- Magnetizing current causes the ratio of winding currents to differ from the turns ratio
Transformer saturation

- Saturation occurs when core flux density $B(t)$ exceeds saturation flux density $B_{sat}$.
- When core saturates, the magnetizing current becomes large, the impedance of the magnetizing inductance becomes small, and the windings are effectively shorted out.
- Large winding currents $i_1(t)$ and $i_2(t)$ do not necessarily lead to saturation. If
  
  \[ 0 = n_1 i_1 + n_2 i_2 \]

  then the magnetizing current is zero, and there is no net magnetization of the core.
- Saturation is caused by excessive applied volt-seconds

---

Saturation vs. applied volt-seconds

Magnetizing current depends on the integral of the applied winding voltage:

\[ i_{M}(t) = \frac{1}{L_M} \int v(t) dt \]

Flux density is proportional:

\[ B(t) = \frac{1}{n_1 A_v} \int v(t) dt \]

Flux density becomes large, and core saturates, when the applied volt-seconds $\lambda_1$ are too large, where

\[ \lambda_1 = \int_{t_1}^{t_2} v(t) dt \]

limits of integration chosen to coincide with positive portion of applied voltage waveform
13.2.3 Leakage inductances

Transformer model, including leakage inductance

Terminal equations can be written in the form

\[
\begin{bmatrix}
  v_1(t) \\
  v_2(t)
\end{bmatrix} = \begin{bmatrix}
  L_{11} & L_{12} \\
  L_{21} & L_{22}
\end{bmatrix} \begin{bmatrix}
  i_1(t) \\
  i_2(t)
\end{bmatrix}
\]

Mutual inductance:

\[ L_{12} = \frac{n_1 n_2}{A} - \frac{n_2^2}{n_1^2} L_M \]

Primary and secondary self-inductances:

\[
\begin{align*}
  L_{11} &= L_{11} + \frac{n_1}{n_2} L_{12} \\
  L_{22} &= L_{22} + \frac{n_2}{n_1} L_{12}
\end{align*}
\]

Effective turns ratio

\[ n_e = \sqrt{\frac{L_{22}}{L_{11}}} \]

Coupling coefficient

\[ k = \frac{L_{12}}{\sqrt{L_{11} L_{22}}} \]
13.3 Loss mechanisms in magnetic devices

Low-frequency losses:
  Dc copper loss
  Core loss: hysteresis loss
High-frequency losses: the skin effect
  Core loss: classical eddy current losses
  Eddy current losses in ferrite cores
High frequency copper loss: the proximity effect
  Proximity effect: high frequency limit
  MMF diagrams, losses in a layer, and losses in basic multilayer windings
Effect of PWM waveform harmonics

13.3.1 Core loss

Energy per cycle \( W \) flowing into \( n \)-turn winding of an inductor, excited by periodic waveforms of frequency \( f \):

\[
W = \int_{\text{one cycle}} v(t) i(t) dt
\]

Relate winding voltage and current to core \( B \) and \( H \) via Faraday’s law and Ampere’s law:

\[
v(t) = nA \frac{dB(t)}{dt}, \quad H(t)\mu_m = n(t)
\]

Substitute into integral:

\[
W = \int_{\text{one cycle}} \left( nA \frac{dB(t)}{dt} \right) \left( \frac{H(t)\mu_m}{n} \right) dt
\]

\[
= \left( A\mu_m \right) \int_{\text{one cycle}} H dB
\]

Core area \( A \)
Core permeability \( \mu_m \)
Core loss: Hysteresis loss

\[ W = \left( A \cdot L_m \right) \int_{\text{one cycle}} H \, dB \]

The term \( A \cdot L_m \) is the volume of the core, while the integral is the area of the \( B-H \) loop.

\[ P_H = \left( f \right) \left( A \cdot L_m \right) \int_{\text{one cycle}} H \, dB \]

Hysteresis loss is directly proportional to applied frequency

Modeling hysteresis loss

- Hysteresis loss varies directly with applied frequency
- Dependence on maximum flux density: how does area of \( B-H \) loop depend on maximum flux density (and on applied waveforms)?
  Empirical equation (Steinmetz equation):
  \[ P_n = K_n f B_{\text{max}}^\alpha (\text{core volume}) \]
  The parameters \( K_n \) and \( \alpha \) are determined experimentally.
  Dependence of \( P_n \) on \( B_{\text{max}} \) is predicted by the theory of magnetic domains.
Core loss: eddy current loss

Magnetic core materials are reasonably good conductors of electric current. Hence, according to Lenz’s law, magnetic fields within the core induce currents ("eddy currents") to flow within the core. The eddy currents flow such that they tend to generate a flux which opposes changes in the core flux $\Phi(t)$. The eddy currents tend to prevent flux from penetrating the core.

Modeling eddy current loss

- Ac flux $\Phi(t)$ induces voltage $v(t)$ in core, according to Faraday’s law. Induced voltage is proportional to derivative of $\Phi(t)$. In consequence, magnitude of induced voltage is directly proportional to excitation frequency $f$.
- If core material impedance $Z$ is purely resistive and independent of frequency, $Z = R$, then eddy current magnitude is proportional to voltage: $i(t) = v(t)/R$. Hence magnitude of $i(t)$ is directly proportional to excitation frequency $f$.
- Eddy current power loss $i(t)R$ then varies with square of excitation frequency $f$.
- Classical Steinmetz equation for eddy current loss:
  $$P_L = K_f f^2 B_m^2 \text{(core volume)}$$
- Ferrite core material impedance is capacitive. This causes eddy current power loss to increase as $f^4$. 
Total core loss: manufacturer’s data

Empirical equation, at a fixed frequency:

\[ P_{fe} = K_{fe} (\Delta B)^p A_e \mu_m \]

Core materials

<table>
<thead>
<tr>
<th>Core type</th>
<th>( B_{sat} )</th>
<th>Relative core loss</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laminations iron, silicon steel</td>
<td>1.5 - 2.0 T</td>
<td>high</td>
<td>50-60 Hz transformers, inductors</td>
</tr>
<tr>
<td>Powdered cores powdered iron, molypermalloy</td>
<td>0.6 - 0.8 T</td>
<td>medium</td>
<td>1 kHz transformers, 100 kHz filter inductors</td>
</tr>
<tr>
<td>Ferrite Manganese-zinc, Nickel-zinc</td>
<td>0.25 - 0.5 T</td>
<td>low</td>
<td>20 kHz - 1 MHz transformers, ac inductors</td>
</tr>
</tbody>
</table>
13.3.2 Low-frequency copper loss

DC resistance of wire

\[ R = \rho \frac{L}{A_w} \]

where \( A_w \) is the wire bare cross-sectional area, and \( L \) is the length of the wire. The resistivity \( \rho \) is equal to \( 1.724 \times 10^{-6} \, \Omega \cdot \text{cm} \) for soft-annealed copper at room temperature. This resistivity increases to \( 2.3 \times 10^{-5} \, \Omega \cdot \text{cm} \) at 100°C.

The wire resistance leads to a power loss of

\[ P_{cu} = I_{rms}^2 R \]

13.4 Eddy currents in winding conductors
13.4.1 Intro to the skin and proximity effects
Penetration depth $\delta$

For sinusoidal currents: current density is an exponentially decaying function of distance into the conductor, with characteristic length $\delta$ known as the penetration depth or skin depth.

$$\delta = \sqrt{\frac{\rho}{\pi \mu f}}$$

For copper at room temperature:

$$\delta = \frac{7.5}{\sqrt{f}} \text{ cm}$$