The Low Q Approximation

Given a second-order denominator polynomial, of the form

\[ G(s) = \frac{1}{1 + a_1 s + a_2 s^2} \quad \text{or} \quad G(s) = \frac{1}{1 + \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \]

When the roots are real, i.e., when \( Q < 0.5 \), then we can factor the denominator, and construct the Bode diagram using the asymptotes for real poles. We would then use the following normalized form:

\[ G(s) = \frac{1}{1 + \frac{s}{\omega_1} \left(1 + \frac{s}{\omega_2}\right)} \]

This is a particularly desirable approach when \( Q < 0.5 \), i.e., when the corner frequencies \( \omega_1 \) and \( \omega_2 \) are well separated.

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L-C-R Example

A problem with this procedure is the complexity of the quadratic formula used to find the corner frequencies.

R-L-C network example:

\[ G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + \frac{s}{R} + s^2 LC} \]

Use quadratic formula to factor denominator. Corner frequencies are:

\[ \omega_1, \omega_2 = \frac{L/R}{2} \pm \sqrt{\left(\frac{L/R}{2}\right)^2 - 4 LC} \]

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Factoring the Denominator

\[ \omega_1, \omega_2 = \frac{L/R \pm \sqrt{[L/R]^2 - 4LC}}{2LC} \]

This complicated expression yields little insight into how the corner frequencies \( \omega_1 \) and \( \omega_2 \) depend on \( R, L, \) and \( C. \)

When the corner frequencies are well separated in value, it can be shown that they are given by the much simpler (approximate) expressions

\[ \omega_1 = \frac{R}{L}, \quad \omega_2 = \frac{1}{RC} \]

\( \omega_1 \) is then independent of \( C, \) and \( \omega_2 \) is independent of \( L. \)

These simpler expressions can be derived via the **Low-\( Q \) Approximation**.

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Derivation of Low-\( Q \) Approximation

Given

\[ G(s) = \frac{1}{1 + \frac{s}{Q \omega_0} + \left( \frac{s}{Q \omega_0} \right)^2} \]

Use quadratic formula to express corner frequencies \( \omega_1 \) and \( \omega_2 \) in terms of \( Q \) and \( \omega_0 \) as:

\[ \omega_1 \approx \frac{\omega_0}{Q} \left( 1 - \sqrt{1 - \frac{4Q^2}{a} \frac{1}{Q \omega_0} \left( 1 - \frac{Q^2}{a} \frac{1}{Q \omega_0} \right)} \right) \]

\[ \omega_2 \approx \frac{\omega_0}{Q} \left( 1 + \sqrt{1 - \frac{4Q^2}{a} \frac{1}{Q \omega_0} \left( 1 - \frac{Q^2}{a} \frac{1}{Q \omega_0} \right)} \right) \]

\[ \omega_3 \approx \frac{\omega_0}{a} \left( 1 + \frac{Q^2}{a} \frac{1}{Q \omega_0} \right) \]

\[ \omega_4 \approx \frac{\omega_0}{a} \]
Corner Frequency $\omega_1$

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

$$\omega_1 = \frac{Q \omega_0}{F(Q)}$$

where

$$F(Q) = \frac{1}{2} \left[ 1 + \sqrt{1 - 4Q^2} \right]$$

For small $Q$, $F(Q)$ tends to 1.

We then obtain

$$\omega_1 = \frac{Q \omega_0}{Q} \text{ for } Q << \frac{1}{2}$$

In practice, apply for $Q \leq \frac{1}{2}$

For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.

---

Corner Frequency $\omega_2$

$$\omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

$$\omega_2 = \frac{Q \omega_0}{F(Q)}$$

where

$$F(Q) = \frac{1}{2} \left[ 1 + \sqrt{1 - 4Q^2} \right]$$

For small $Q$, $F(Q)$ tends to 1.

We then obtain

$$\omega_2 = \frac{Q \omega_0}{Q} \text{ for } Q << \frac{1}{2}$$

For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.
Low-Q Approximation Results

For the previous example:

\[ G(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + \frac{1}{R} + s^2LC} \]

\[ f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \]

\[ Q = R\sqrt{\frac{C}{L}} \]

Use of the Low-Q Approximation leads to

\[ \omega_0 = Q \omega_0 = R \sqrt{\frac{C}{L}} \cdot \frac{1}{\sqrt{LC}} = \frac{R}{L} \]

\[ \frac{\omega_0}{Q} = \sqrt{\frac{C}{L}} \cdot \frac{1}{R \sqrt{C}} = \frac{1}{RC} \]

The Low-Q Approximation

\[ ||G||_{dB} \]

\[ f_1 = \frac{Qf_0}{F(Q)} \]

\[ f_0 = \frac{Qf_0}{Q} \]

\[ f_2 = \frac{f_0F(Q)}{Q} \]

-20dB/decade

-40dB/decade
**Example: Damped Input EMI Filter**

\[ G(s) = \frac{i_d(s)}{i_i(s)} = \frac{1 + \frac{L_1 + L_2}{R}}{1 + s\frac{L_1 + L_2}{R} + \frac{s^2 L_1 C + s^3 L_1 L_2 C}{R}} \]

Denominator factors into \((1 + \tau_1 s)(1 + \tau_2 s)(1 + \tau_3 s)\) or \((1 + \frac{1}{\tau_1})s(1 + \frac{1}{\tau_2})s(1 + \frac{1}{\tau_3})s\)

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**8.1.8: Approximate Roots of a Polynomial**

Generalize the low-Q approximation to obtain approximate factorization of the \(n\)-th order polynomial

\[ P(s) = 1 + a_1 s + a_2 s^2 + \cdots + a_n s^n \]

It is desired to factor this polynomial in the form

\[ P(s) = \left(1 + \tau_1 s\right)\left(1 + \tau_2 s\right)\cdots\left(1 + \tau_n s\right) \]

When the roots are real and well separated in value, then approximate analytical expressions for the time constants \(\tau_1, \tau_2, \ldots, \tau_n\) can be found, that typically are simple functions of the circuit element values.

**Objective:** find a general method for deriving such expressions. Include the case of complex root pairs.
Derivation of the Approximation

Multiply out factored form of polynomial, then equate to original form (equate like powers of $i$):

\[ a_i = r_1 + r_2 + \cdots + r_s \]
\[ a_2 = r_1(r_2 + \cdots + r_s) + r_2(r_1 + \cdots + r_s) + \cdots \]
\[ a_3 = r_1r_2(r_3 + \cdots + r_s) + r_3(r_1r_2 + \cdots + r_s) + \cdots \]
\[ \vdots \]
\[ a_s = r_1r_2\cdots r_s \]

- Exact system of equations relating roots to original coefficients
- Exact general solution is hopeless
- Under what conditions can solution for time constants be easily approximated?

Case When All Roots Separate

System of equations:

(From previous slide)

\[ a_i = r_1 + r_2 + \cdots + r_s \approx \tau_i \]
\[ a_2 = r_1(r_2 + \cdots + r_s) + r_2(r_1 + \cdots + r_s) + \cdots \approx \tau_1\tau_2 \]
\[ a_3 = r_1r_2(r_3 + \cdots + r_s) + r_3(r_1r_2 + \cdots + r_s) + \cdots \approx \tau_1\tau_2\tau_3 \]
\[ \vdots \]
\[ a_s = r_1r_2\cdots r_s \]

Suppose that roots are real and well-separated, and are arranged in decreasing order of magnitude:

\[ |\tau_1| >> |\tau_2| >> \cdots >> |\tau_s| \]

Then the first term of each equation is dominant

\[ \Rightarrow \text{Neglect second and following terms in each equation above} \]
Approximation When Roots are Well Separated

System of equations:
(only first term in each equation is included)

\[ a_1 = \tau_1 \]
\[ a_2 = \tau_1 \tau_2 \]
\[ a_3 = \tau_1 \tau_2 \tau_3 \]
\[ \vdots \]
\[ a_n = \tau_1 \tau_2 \tau_3 \cdots \tau_n \]

Solve for the time constants:

\[ \tau_1 = a_1 \]
\[ \tau_2 = \frac{a_2}{a_1} \]
\[ \tau_3 = \frac{a_3}{a_2} \]
\[ \vdots \]
\[ \tau_n = \frac{a_n}{a_{n-1}} \]

Results

If the following inequalities are satisfied

\[ a_1 \gg \frac{a_2}{a_1} \gg \frac{a_3}{a_2} \gg \cdots \gg \frac{a_n}{a_{n-1}} \]

Then the polynomial \( P(s) \) has the following approximate factorization

\[ P(s) = \left(1 + \frac{a_1}{s}\right) \left(1 + \frac{a_2}{a_1} s\right) \left(1 + \frac{a_3}{a_2} \frac{a_1}{s}\right) \cdots \left(1 + \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_2}{a_1} \frac{a_1}{s}\right) \]

- If the \( a_n \) coefficients are simple analytical functions of the element values \( L, C, \) etc., then the roots are similar simple analytical functions of \( L, C, \) etc.
- **Numerical values** are used to justify the approximation, but analytical expressions for the roots are obtained
Quadratic Roots: Not Well Separated

Suppose inequality $i$ is not satisfied:

$$|a_i| > |\frac{a_2}{a_1}| > ... > |\frac{a_k}{a_{k-1}}| > |\frac{a_{k+1}}{a_k}| > ... > |\frac{a_n}{a_{n-1}}|$$

Then leave the terms corresponding to roots $i$ and $(k+1)$ in quadratic form, as follows:

$$P(s) = \left(1 + a_i s \right) \left(1 + \frac{a_2}{a_1} s \right) \ldots \left(1 + \frac{a_k}{a_{k-1}} s + \frac{a_{k+1}}{a_k} s^2 \right) \ldots \left(1 + \frac{a_n}{a_{n-1}} s \right)$$

This approximation is accurate provided

$$|a_i| > |\frac{a_2}{a_1}| > ... > |\frac{a_k}{a_{k-1}}| > |\frac{a_{k+1}}{a_k}| > ... > |\frac{a_n}{a_{n-1}}|$$

First Inequality Violated

When inequality 1 is not satisfied:

$$|a_i| > |\frac{a_2}{a_1}| > ... > |\frac{a_k}{a_{k-1}}| > |\frac{a_{k+1}}{a_k}| > ... > |\frac{a_n}{a_{n-1}}|$$

Then leave the first two roots in quadratic form, as follows:

$$P(s) = \left(1 + a_i s + a_2 s^2 \right) \left(1 + \frac{a_3}{a_2} s \right) \ldots \left(1 + \frac{a_{k+1}}{a_k} s + \frac{a_{k+2}}{a_{k+1}} s^2 \right) \ldots$$

This approximation is justified provided

$$|\frac{a_2}{a_1}| > |a_i| > |\frac{a_3}{a_2}| > ... > |\frac{a_{k+1}}{a_k}| > ... > |\frac{a_n}{a_{n-1}}|$$
Other Cases

• Several nonadjacent inequalities violated
  – Apply same process multiple times
• Multiple adjacent inequalities violated
  – More than two roots close in value
  – Must use 3rd order or higher polynomial

8.2 Analysis of Converter TFs

8.2.1. Example: transfer functions of the buck-boost converter
8.2.2. Transfer functions of some basic CCM converters
8.2.3. Physical origins of the right half-plane zero in converters
Example: Buck-Boost

Small-signal ac model of the buck-boost converter, derived in Chapter 7:

Line-to-Output TF

\[ G_{vq} = \frac{v_{q}}{i_{q}} \]

An alternative independent input defined \((q - q')\)

\[ V = -v_{q} \left( \frac{RL_{se}}{R(1+SCR)L_{se}} + \frac{sL_{se}}{s} \right) \]

\[ \frac{dL}{ds} \frac{1}{i_{c}} = \frac{B}{1+SCR} \]

\[ G_{vq} = \frac{v_{q}}{i_{q}} \]

\[ G_{vq} = G_{vo} \frac{1}{1 + \frac{L}{C}s + \frac{sL}{o}} \]

\[ G_{vq} = G_{vo} \frac{1}{s} \]

\[ G_{vq} = G_{vo} \frac{1+ \frac{L}{C}s + \frac{sL}{o}}{s} \]

\[ Q = \frac{dQ}{ds} = 0 \]

\[ \omega_{o} = \frac{D}{L} \]

\[ \omega_{o} = \frac{D}{2L} \]

\[ \omega_{o} = \frac{D}{12L} \]

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