G-S Power Flow Solution

\[ I_i = \frac{P_i - jQ_i}{V_i^*} = V_i \sum_{j=0}^{n} y_{ij} - \sum_{j=1, j \neq i}^{n} y_{ij} V_j \]

\[ Y_{ij} = -y_{ij} \quad j \neq i \]

\[ Y_{ii} = \sum_{j=0}^{n} y_{ij} \]

\[ \begin{align*}
V_i^{(k+1)} &= \frac{P_i^{(k)} - jQ_i^{(k)}}{V_i^{* (k)}} - \sum_{j=1, j \neq i}^{n} Y_{ij} V_j^{(k)} \\
Q_i^{(k+1)} &= -\Im[V_i^{* (k)} \sum_{j=1}^{n} Y_{ij} V_j^{(k)}] \\
P_i^{(k+1)} &= \Re[V_i^{* (k)} \sum_{j=1}^{n} Y_{ij} V_j^{(k)}]
\end{align*} \]
Calculation for PQ Buses

- $|V_i|$ and $\delta_i$ are unknown
- $P_i$ and $Q_i$ are scheduled (generation/load), denoted by $P_i^{sch}$ and $Q_i^{sch}$

$$x^{(k+1)} = g(x^{(k)})$$

$$V_i^{(k+1)} = \frac{P_i^{sch} - jQ_i^{sch}}{V_i^{*(k)}} - \sum_{j=1,j \neq i}^n Y_{ij} V_j^{(k)}$$

- Under normal operating conditions:
  - Slack bus: $|V_0| \angle \delta_0$ (typically $1 \angle 0^\circ$)
  - Other buses: $|V_i|$ is close to 1pu or $|V_0|$. For most of cases, there are:
    - Generator buses: $|V_i| > |V_0|$, $\delta_i > \delta_0$,
    - Load buses: $|V_i| < |V_0|$, $\delta_i < \delta_0$

- Initial guess could be $V_i^{(0)} = 1 \angle 0^\circ$ if a better estimate is unavailable.
Calculation for PV Buses

- $P_i = P_i^{sch}$ and $|V_i|$ are specified
- Starting from an initial estimate of $\delta_i^{(0)} \rightarrow V_i^{(0)} = |V_i| \leq \delta_i^{(0)}$

$$Q_i^{(k+1)} = -\text{Im}[V_i^{*(k)} \sum_{j=1}^{n} Y_{ij} V_j^{(k)}]$$

$$V^{(k+1)}_{ci} = \frac{P_i^{sch} - jQ_i^{(k+1)}}{V_i^{*(k)}} - \sum_{j=1, j\neq i}^{n} \frac{Y_{ij} V_j^{(k)}}{Y_{ii}}$$

- Since $|V_i|$ is specified, only $V_{I,i}^{(k+1)} = \text{Im}[V_{ci}^{(k+1)}]$ is retained

$$V_{R,i}^{(k+1)} = \sqrt{|V_i|^2 - (V_{I,i}^{(k+1)})^2}$$

Update $V_i^{(k+1)} = V_{R,i}^{(k+1)} + j \cdot V_{I,i}^{(k+1)}$

- Continue the iterations until

$$|V_{R,i}^{(k+1)} - V_{R,i}^{(k)}| \leq \varepsilon \quad |V_{I,i}^{(k+1)} - V_{I,i}^{(k)}| \leq \varepsilon$$

or, the power mismatch, i.e. the largest element in $\Delta P$ and $\Delta Q < \varepsilon$

- Using acceleration factor $\alpha = 1.3$ to 1.7

$$V_{ic}^{(k+1)} = V_i^{(k)} + \alpha \left( \frac{P_i^{sch} - jQ_i^{(k+1)}}{V_i^{*(k)}} - \sum_{j=1, j\neq i}^{n} \frac{Y_{ij} V_j^{(k)}}{Y_{ii}} - V_i^{(k)} \right)$$
Calculation for Slack Bus

\[ P_i^{(k+1)} - jQ_i^{(k+1)} = V_i^{*(k+1)} \sum_{j=1}^{n} Y_{ij} V_j^{(k+1)} \]

Calculation of Line Flows and Losses

• At bus \( i \):
\[ I_{ij} = I_l + I_{i0} = y_{ij} (V_i - V_j) + y_{i0} V_i \]
\[ S_{ij} = V_i I_{ij}^{*} \]

• At bus \( j \):
\[ I_{ji} = -I_l + I_{j0} = y_{ij} (V_j - V_i) + y_{j0} V_j \]
\[ S_{ji} = V_j I_{ji}^{*} \]

• Power loss in line \( i - j \):
\[ S_{Lij} = S_{ij} + S_{ji} \]
Example 6.7
(slack bus + 2 P-Q buses)

Using the G-S method to find the power flow solution:

(a) Determine the voltage phasors at P-Q buses 2 and 3 accurate to 4 decimal places

(b) Find the slack bus real and reactive power

(c) Determine the line flows and losses. Show line flow directions in a power-flow diagram

(Solve \( P_1, Q_1, \ |V_2|, \delta_2, |V_3|, \delta_3, \ S_{ij} \) and \( S_{lij} \))

**Step 1. Check what are known**

(a) Line impedances are converted to admittances

\[ y_{12} = \frac{1}{0.02 + j0.04} = 10 - j20 \]

Similarly, \( y_{13} = 10 - j30 \) and \( y_{23} = 16 - j32 \).

At the P-Q buses, the complex loads expressed in per units are

\[ S_{2}^{sch} = \frac{(256.6 + j110.2)}{100} = -2.566 - j1.102 \text{ pu} \]
\[ S_{3}^{sch} = \frac{(138.6 + j45.2)}{100} = -1.386 - j0.452 \text{ pu} \]

**Step 2. Set initial estimates and start to iterate**

Starting from an initial estimate of \( V_2^{(0)} = 1.0 + j0.0 \)

and \( V_3^{(0)} = 1.0 + j0.0 \),

\[ V_2^{(1)} = \frac{P_2^{sch} - jQ_2^{sch}}{V_2^{(0)}} + \frac{y_{12}V_1 + y_{23}V_3^{(0)}}{y_{12} + y_{23}} = 0.9825 - j0.0310 \]

\[ V_3^{(1)} = \frac{P_3^{sch} - jQ_3^{sch}}{V_3^{(0)}} + \frac{y_{13}V_1 + y_{23}V_2^{(1)}}{y_{13} + y_{23}} = 1.0011 - j0.0353 \]

\[ V_2^{(2)} = 0.9816 - j0.0520 \quad V_3^{(2)} = 1.0008 - j0.0459 \]

\[ V_2^{(6)} = 0.9801 - j0.0599 \quad V_3^{(6)} = 1.0000 - j0.0500 \]

\[ V_2^{(7)} = 0.9800 - j0.0600 \quad V_3^{(7)} = 1.0000 - j0.0500 \]

The final solution is

\[ V_2 = 0.9800 - j0.0600 = 0.98183 \angle -3.5035^\circ \text{ pu} \]
\[ V_3 = 1.0000 - j0.0500 = 1.00125 \angle -2.8624^\circ \text{ pu} \]
Step 3. Calculate P and Q of the slack bus

(b) With the knowledge of all bus voltages, the slack bus power is

$$P_1 - jQ_1 = V_1^* [V_1 (y_{12} + y_{13}) - (y_{12} V_2 + y_{13} V_3)] = 4.095 - j1.890$$

the slack bus real and reactive powers are $P_1 = 4.095$ pu = 409.5 MW and $Q_1 = 1.890$ pu = 189 Mvar.

Step 4. Calculate line currents, flows and losses

$$I_{ij} = y_{ij} (V_i - V_j) + y_{i0} V_i$$
$$I_{ji} = y_{ij} (V_j - V_i) + y_{j0} V_j$$

$$I_{12} = y_{12} (V_1 - V_2) = (10 - j20) [(1.05 + j0) - (0.98 - j0.06)] = 1.9 - j0.8$$
$$I_{21} = -I_{12} = -1.9 + j0.8$$
$$I_{13} = y_{13} (V_1 - V_3) = (10 - j30) [(1.05 + j0) - (1.0 - j0.05)] = 2.0 - j1.0$$
$$I_{31} = -I_{13} = -2.0 + j1.0$$
$$I_{23} = y_{23} (V_2 - V_3) = (16 - j32) [(0.98 - j0.06) - (1 - j0.05)] = -0.64 + j4.48$$
$$I_{32} = -I_{23} = 0.64 - j0.48$$

$$S_{ji} = V_j I_{ji}^*$$
$$S_{ij} = V_i I_{ij}^*$$

$$S_{12} = 199.5 \text{ MW} + j84.0 \text{ Mvar}$$
$$S_{31} = V_3 I_{31}^* = -205.0 \text{ MW} - j90.0 \text{ Mvar}$$
$$S_{21} = -191.0 \text{ MW} - j67.0 \text{ Mvar}$$
$$S_{23} = V_2 I_{23}^* = -65.6 \text{ MW} - j43.2 \text{ Mvar}$$
$$S_{13} = 210.0 \text{ MW} + j105.0 \text{ Mvar}$$
$$S_{32} = V_3 I_{32}^* = 66.4 \text{ MW} + j44.8 \text{ Mvar}$$

$$S_{L12} = S_{12} + S_{21} = 8.5 \text{ MW} + j17.0 \text{ Mvar}$$
$$S_{L13} = S_{13} + S_{31} = 5.0 \text{ MW} + j15.0 \text{ Mvar}$$
$$S_{L23} = S_{23} + S_{32} = 0.8 \text{ MW} + j1.60 \text{ Mvar}$$

**FIGURE 6.11**

Power flow diagram of Example 6.7 (powers in MW and Mvar).
Example 6.8
(slack bus + P-Q bus + P-V bus)

Line charging susceptances are neglected. Obtain the power flow solution by the G-S method including line flows and line losses (Solve \( P_1, Q_1, |V_2|, \delta_2, Q_3, \delta_3, S_{ij} \) and \( S_{lij} \))

**Step 1. Check what are known**

\[
S_{2sch} = \frac{-(400 + j250)}{100} = -4.0 - j2.5 \text{ pu}
\]

\[
P_{3sch} = \frac{200}{100} = 2.0 \text{ pu}
\]

**Step 2. Set initial estimates and start to iterate**

Starting from an initial estimate of \( V_2^{(0)} = 1.0 + j0.0 \) and \( V_3^{(0)} = 1.04 + j0.0 \),

\[
V_i^{(k+1)} = \frac{P_i^{sch} - jQ_i^{sch}}{V_i^{*(k)}} - \sum_{j=1, j \neq i}^n Y_{ij}V_j^{(k)}
\]

\[
V_2^{(1)} = \frac{P_2^{sch} - jQ_2^{sch}}{V_2^{(0)}} + y_{12}V_1 + y_{23}V_3^{(0)}
\]

\[
= \frac{0.02 + j0.04}{0.01 + j0.03} + 10 - j20 = 0.97462 - j0.042307
\]

\[
Q_i^{(k+1)} = -\text{Im}[V_i^{*(k)}\sum_{j=1}^n Y_{ij}V_j^{(k)}]
\]

\[
Q_3^{(1)} = -\Re\{V_3^{*(0)}[V_3^{(0)}(y_{13} + y_{23}) - y_{13}V_1 - y_{23}V_2^{(1)}]\} = 1.16
\]

Note: \(|V_{c3}^{(1)}|=1.0378 \neq 1.04=|V_3|

Since \(|V_3|\) is held constant at 1.04 pu, only the imaginary part of \( V_{c3}^{(1)} \) is retained, and its real part is obtained from

\[
V_R^{(1)} = \sqrt{(1.04)^2 - (0.005170)^2} = 1.039987
\]

\[
V_3^{(1)} = 1.039987 - j0.005170
\]
Bus 2 (P-Q): Solve $V_2$, $\delta_2$

$$V_i^{(k+1)} = \frac{P_i^{sch} - jQ_i^{sch}}{V_i^{(k)}} - \sum_{j=1, j \neq i}^n \frac{Y_{ij}V_j^{(k)}}{Y_{ii}}$$

$$V_2^{(2)} = 0.971057 - j0.043432$$
$$V_2^{(3)} = 0.97073 - j0.04479$$
$$V_2^{(4)} = 0.97065 - j0.04533$$
$$V_2^{(5)} = 0.97062 - j0.04555$$
$$V_2^{(6)} = 0.97061 - j0.04565$$
$$V_2^{(7)} = 0.97061 - j0.04569$$

The final solution is

$$V_2 = 0.97168 \angle -2.6948^\circ \text{ pu}$$

$$S_3 = 2.0 + j1.4617 \text{ pu}$$

$$V_3 = 1.04 \angle -0.498^\circ \text{ pu}$$

$$P_i - jQ_i = V_i^* \sum_{j=1}^n Y_{ij}V_j$$

Bus 3 (P-V): Solve $Q_3$, $\delta_3$

$$Q_i^{(k+1)} = -\text{Im}[V_i^* \sum_{j=1, j \neq i}^n Y_{ij}V_j^{(k)}]$$

$$V_i^{(k+1)} = \frac{P_i^{sch} - jQ_i^{(k+1)}}{V_i^{(k)}} - \sum_{j=1, j \neq i}^n \frac{Y_{ij}V_j^{(k)}}{Y_{ii}}$$

$$Q_3^{(2)} = 1.38796$$
$$Q_3^{(3)} = 1.42904$$
$$Q_3^{(4)} = 1.44833$$
$$Q_3^{(5)} = 1.45621$$
$$Q_3^{(6)} = 1.45947$$
$$Q_3^{(7)} = 1.46082$$

$$V_{c3}^{(2)} = 1.03908 - j0.00730$$
$$V_{c3}^{(3)} = 1.03954 - j0.00833$$
$$V_{c3}^{(4)} = 1.03978 - j0.00873$$
$$V_{c3}^{(5)} = 1.03989 - j0.00900$$
$$V_{c3}^{(6)} = 1.03993 - j0.00900$$
$$V_{c3}^{(7)} = 1.03995 - j0.00903$$

Re$[V_{c3}^{(k+1)}] = \sqrt{1.04^2 - \text{Im}[V_{c3}^{(k+1)}]^2}$

The final solution is

$$V_3 = 1.039974 - j0.00730 \text{ pu}$$

$$S_3 = 2.0 + j1.4617 \text{ pu}$$

$$V_3 = 1.04 \angle -0.498^\circ \text{ pu}$$

Bus 1 (V-\delta): $P_1$, $Q_1$

$$S_1 = 2.1842 + j1.4085 \text{ pu}$$

$$S_{12} = 179.36 + j118.734$$
$$S_{21} = -170.97 - j101.947$$
$$S_{13} = 39.06 + j22.118$$
$$S_{31} = -38.88 - j21.569$$
$$S_{23} = -229.03 - j148.05$$
$$S_{32} = 238.88 + j167.746$$

$$S_{L12} = 8.39 + j16.79$$
$$S_{L13} = 0.18 + j0.548$$
$$S_{L23} = 9.85 + j19.69$$
Tap Changing Transformers

- $a$ is the per unit off-nominal tap position (usually, $|a| = 0.9$ to $1.1$)
  - Complex number for phase shifting transformers

\[
S_T = V_x I_i^* = -V_j I_j^* \quad V_x = \frac{1}{a} V_j \quad \Rightarrow \quad I_i = -a^* \cdot I_j
\]

\[
I_i = y_t (V_i - V_x) = y_t V_i - \frac{y_t}{a} V_j
\]

\[
I_j = -\frac{1}{a^*} I_i = -\frac{y_t}{a} V_i + \frac{y_t}{|a|^2} V_j
\]

$$
\begin{bmatrix}
I_i \\
I_j
\end{bmatrix} =
\begin{bmatrix}
y_t & -\frac{y_t}{a} \\
-\frac{y_t}{a^*} & \frac{y_t}{|a|^2}
\end{bmatrix}
\begin{bmatrix}
V_i \\
V_j
\end{bmatrix}
$$

$Y_{bus}$ is not symmetrical if a phase shifting transformer exists in the system

Equivalent circuit if $a$ is real (ignoring phase shifting)

Non-tap side

Tap side
Newton-Raphson Method

- Based on Taylor’s series expansion at an initial estimate of the solution

\[ f(x) = c \]

\[ f(x^{(0)} + \Delta x^{(0)}) = c \]

\[ f(x^{(0)}) + \frac{df}{dx}^{(0)} \Delta x^{(0)} = c \]

- Ignore all terms with orders \( \geq 2 \)

Comparison: G-S method ignores all differential terms (orders \( \geq 1 \))

\[ \frac{df}{dx}^{(0)} \Delta x^{(0)} = c - f(x^{(0)}) \Delta c^{(0)} \]

\[ \Delta x^{(0)} = \frac{\Delta c^{(0)}}{\frac{df}{dx}^{(0)}} \]

\[ x^{(1)} = x^{(0)} + \Delta x^{(0)} \]
• Iteration 1: (0) → (1)

\[ x^{(1)} = x^{(0)} + \Delta x^{(0)} = x^{(0)} + \frac{\Delta c^{(0)}}{(\frac{df}{dx})^{(0)}} = x^{(0)} + \frac{c - f(x^{(0)})}{(\frac{df}{dx})^{(0)}} \]

• Iteration k+1: (k) → (k+1)

\[
\Delta x^{(k)} = \frac{\Delta c^{(k)}}{(\frac{df}{dx})^{(k)}}
\]

• Until \( |x^{(k+1)} - x^{(k)}| \leq \varepsilon \)

• \( f(x) = c \) is actually approximated by its tangent line at \( x = x^{(k)} \).

\[
(\frac{df}{dx})^{(k)} (x - x^{(k)}) + f(x^{(k)}) = c
\]
Example 6.4

Let $x^{(0)}=6$

$$\frac{df}{dx}(x) = 3x^2 - 12x + 9$$

$$(\frac{df}{dx})^{(0)} = 3(6)^2 - 12(6) + 9 = 45$$

$$\Delta c^{(0)} = c - f(x^{(0)}) = 0 - [(6)^3 - 6(6)^2 + 9(6) - 4] = -50$$

$$\Delta x^{(0)} = \frac{\Delta c^{(0)}}{(\frac{df}{dx})^{(0)}} = \frac{-50}{45} = -1.1111$$

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = 6 - 1.1111 = 4.8889$$

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 4.8889 - \frac{13.4431}{22.037} = 4.2789$$

$$x^{(3)} = x^{(2)} + \Delta x^{(2)} = 4.2789 - \frac{2.9981}{12.5797} = 4.0405$$

$$x^{(4)} = x^{(3)} + \Delta x^{(3)} = 4.0405 - \frac{0.3748}{9.4914} = 4.0011$$

$$x^{(5)} = x^{(4)} + \Delta x^{(4)} = 4.0011 - \frac{0.0095}{9.0126} = 4.0000$$
N-dimensional System

\[ f(x) = c \]

\[ x^{(k+1)} = x^{(k)} + \Delta x^{(k)} = x^{(k)} + \left( \frac{df}{dx} \right)^{(k)} \Delta c^{(k)} \]

\[ \Delta c^{(k)} = c - f(x^{(k)}) \]

\[ f_1(x_1, x_2, \ldots, x_n) = c_1 \]
\[ f_2(x_1, x_2, \ldots, x_n) = c_2 \]
\[ \ldots \]
\[ f_n(x_1, x_2, \ldots, x_n) = c_n \]

Jacobian Matrix:

\[ J^{(k)} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}^{(k)} & \frac{\partial f_1}{\partial x_2}^{(k)} & \cdots & \frac{\partial f_1}{\partial x_n}^{(k)} \\
\frac{\partial f_2}{\partial x_1}^{(k)} & \frac{\partial f_2}{\partial x_2}^{(k)} & \cdots & \frac{\partial f_2}{\partial x_n}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1}^{(k)} & \frac{\partial f_n}{\partial x_2}^{(k)} & \cdots & \frac{\partial f_n}{\partial x_n}^{(k)}
\end{bmatrix} \]

\[ \Delta X^{(k)} = \begin{bmatrix}
\Delta x_1^{(k)} \\
\Delta x_2^{(k)} \\
\vdots \\
\Delta x_n^{(k)}
\end{bmatrix} \]

\[ \Delta C^{(k)} = \begin{bmatrix}
c_1 - (f_1)^{(k)} \\
c_2 - (f_2)^{(k)} \\
\vdots \\
c_n - (f_n)^{(k)}
\end{bmatrix} \]
Example 6.5

- Use the N-R method to find the intersections of the curves

\[ x_1^2 + x_2^2 = 4 \]
\[ e^{x_1} + x_2 = 1 \]

\[ J = \begin{pmatrix} 2x_1 & 2x_2 \\ e^{x_1} & 1 \end{pmatrix} \]

\( J^{(k)} \) tells the fastest direction (gradient) seen from point \( k \) on the path toward a solution

If \( x_1^{(0)}=2, \ x_2^{(0)}=-2 \):

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<th>( k )</th>
<th>( \Delta C )</th>
<th>( J )</th>
<th>( \Delta x )</th>
<th>( x )</th>
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<td>1.0000</td>
<td>-0.0000</td>
</tr>
</tbody>
</table>
Compared to the Gauss-Seidel Method

• Since higher-order terms are ignored, the N-R method also needs the initial estimation to be sufficiently close to the actual solution.

• The N-R method converges much faster
  – N-R method: quadratic convergence (ignoring the 2\textsuperscript{nd} and higher orders)
  – G-S method: linear convergence (ignoring the 1\textsuperscript{st} and higher orders)

• The N-R method has more computational complexity:
  – Requires $[J^{(k)}]^{-1}$ during each iteration, which is computationally intense.
**Dealing with \([J^{(k)}]^{-1}\)**

\[ \Delta X^{(k+1)} = [J^{(k)}]^{-1} \Delta C^{(k)} \]

- Try not to update \(J^{(k)}\) so often (at least not in every iteration)
- Apply LU decomposition (triangular factorization):
  - Instead of calculating \(\Delta X\) directly by \(J^{-1}\), we first solve \((U\Delta X)\) and then solve \(\Delta X\)

\[
A = LU,
\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
= \begin{bmatrix}
l_{11} & 0 & 0 \\
l_{21} & l_{22} & 0 \\
l_{31} & l_{32} & l_{33}
\end{bmatrix}
\begin{bmatrix}
u_{11} & u_{12} & u_{13} \\
u_{22} & u_{22} & u_{23} \\
u_{33}
\end{bmatrix}
\]

\[
J^{(k)} \Delta X^{(k+1)} = L^{(k)} \left( U^{(k)} \Delta X^{(k+1)} \right) = \Delta C^{(k)}
\]

In MATLAB, the solution of \(J\Delta X = \Delta C\) can be obtained by
\[
\Delta X = J \backslash \Delta C
\]