Parameter Optimization with Inequality Constraints

Minimize \( f(x_1, x_2, \ldots, x_n) \)

Subject to:
\[
g_k(x_1, x_2, \ldots, x_n) = 0 \quad k = 1, 2, \ldots, K
\]
\[
u_j(x_1, x_2, \ldots, x_n) \leq 0 \quad j = 1, 2, \ldots, m
\]

- Introduce Lagrange Multipliers \( \lambda_1 \ldots \lambda_K \) and \( \mu_1 \ldots \mu_m \)

\[
L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_K, \mu_1, \ldots, \mu_m) = f + \sum_{k=1}^{K} \lambda_k g_k + \sum_{j=1}^{m} \mu_j u_j
\]

- Necessary conditions for the local minima of \( L \)
\[
\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{k=1}^{K} \lambda_k \frac{\partial g_k}{\partial x_i} + \sum_{j=1}^{m} \mu_j \frac{\partial u_j}{\partial x_i} = 0 \quad i = 1, \ldots, n
\]
\[
\frac{\partial L}{\partial \lambda_k} = g_k = 0 \quad k = 1, 2, \ldots, K
\]
\[
\frac{\partial L}{\partial \mu_j} = u_j \leq 0
\]
\[
\mu_j u_j = 0 \quad \mu_j \geq 0 \quad j = 1, \ldots, m
\]

KKT (Karush–Kuhn–Tucker) necessary conditions

Feasible solutions
• Minimize \( f(x, y) = x^2 + y^2 \)
Subject to \((x-8)^2 + (y-6)^2 = 25 \rightarrow g(x, y) = (x-8)^2 + (y-6)^2 - 25 = 0 \)
\[2x + y \geq 12 \rightarrow u(x, y) = 12 - 2x - y \leq 0\]

\[
L = x^2 + y^2 + \lambda [(x - 8)^2 + (y - 6)^2 - 25] + \mu (12 - 2x - y)
\]

\[
\frac{\partial L}{\partial x} = 2x + \lambda (2x - 16) - 2\mu = 0
\]

\[
\frac{\partial L}{\partial y} = 2y + \lambda (2y - 12) - \mu = 0
\]

\[
\frac{\partial L}{\partial \lambda} = (x - 8)^2 + (y - 6)^2 - 25 = 0
\]

\[
\frac{\partial L}{\partial \mu} \leq 0,
\frac{\partial L}{\partial \mu} = 12 - 2x - y < 0 \quad \mu = 0
\]

\[
\mu_j u_j = 0, \quad \mu_j \geq 0 \quad \text{or} \quad \frac{\partial L}{\partial \mu} = 12 - 2x - y = 0 \quad \mu > 0
\]

Solutions:
\[
\mu = 0, \quad \lambda = -3, \quad x = 12 \text{ and } y = 9 \quad (f = 225)
\]
\[
\mu = -5.6, \quad \lambda = -0.2, \quad x = 5 \quad \text{and } y = 2 \quad (f = 29)
\]
\[
\mu = -12, \quad \lambda = -1.8, \quad x = 3 \quad \text{and } y = 6 \quad (f = 33)
\]
Operating Cost of a Thermal Plant

- Fuel-cost curve of a generator (represented by a quadratic function of real power)
  \[ C_i(P_i) = \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]
- Incremental fuel-cost curve:
  \[ \lambda_i = \frac{dC_i}{dP_i} = 2\gamma_i P_i + \beta_i \]
A real example

<table>
<thead>
<tr>
<th>Gen</th>
<th>ID</th>
<th>PRIOR</th>
<th>FUELCO ($/MBtu)</th>
<th>P_{MAX} (MW)</th>
<th>P_{MIN} (WM)</th>
<th>HE_{MIN} (MBtu/h)</th>
<th>X1 (MW)</th>
<th>Y1 (Btu/kWh)</th>
<th>X2 (MW)</th>
<th>Y2 (Btu/kWh)</th>
<th>X3 (MW)</th>
<th>Y3 (Btu/kWh)</th>
</tr>
</thead>
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<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1.91</td>
<td>230</td>
<td>65</td>
<td>532</td>
<td>65</td>
<td>8760</td>
<td>176</td>
<td>9507</td>
<td>260</td>
<td>10072</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>0</td>
<td>0.539</td>
<td>106</td>
<td>50</td>
<td>425</td>
<td>50</td>
<td>8501</td>
<td>75</td>
<td>9198</td>
<td>106</td>
<td>10341</td>
</tr>
</tbody>
</table>

Fuel input (Btu/h) = \( X_i \times Y_i \times 10^3 \)

Cost ($/h) = Btu/h \times $/MBtu / 10^6

\( \lambda_i ($/MWh) = Y_i /10^3 \times $/MBtu \)

FIGURE 7.3
(a) Heat-rate curve. (b) Fuel-cost curve.

FIGURE 7.4
Typical incremental fuel-cost curve.
Economic Dispatch (ED) Neglecting Losses and No Generator Limits

If transmission line losses are neglected, minimize the total production cost:

\[ C_t = \sum_{i=1}^{n_g} C_i = \sum_{i=1}^{n} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]

subject to

\[ \sum_{i=1}^{n_g} P_i = P_D \]

• Apply the Lagrange multiplier method
  \((n_g + 1)\) unknowns to solve):

\[ L = C_t + \lambda (P_D - \sum_{i=1}^{n_g} P_i) \]

\[ \frac{\partial L}{\partial P_i} = 0 \quad \Rightarrow \quad \frac{\partial C_t}{\partial P_i} - \lambda = 0 \quad \Rightarrow \quad \frac{\partial C_t}{\partial P_i} = \frac{dC_i}{dP_i} = \beta_i + 2\gamma_i P_i = \lambda \]

\[ \frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n_g} P_i = P_D \quad \Rightarrow \quad \sum_{i=1}^{n_g} \frac{\lambda - \beta_i}{2\gamma_i} = P_D \]

\[ \lambda = \frac{P_D + \sum_{i=1}^{n_g} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_g} \frac{1}{2\gamma_i}} \]

All plants must operate at equal incremental cost

\[ P_i = \frac{\lambda - \beta_i}{2\gamma_i} \quad \text{for} \quad i = 1, \ldots, n_g \]

Solve \(P_i\)
Example 7.4

The fuel-cost functions for three thermal plants are $C_1$ to $C_2$ in $$/h. P_1$, $P_2$ and $P_3$ are in MW. $P_D=800$ MW. Neglecting line losses and generator limits, find the optimal dispatch and the total cost in $$/h.

$$P_D + \sum_{i=1}^{n_g} \frac{\beta_i}{2\gamma_i}$$

$$\lambda = \frac{\lambda - \beta_i}{2\gamma_i}$$

$$P_i = \frac{8.5 - 5.3}{2(0.004)} = 400.0000$$

$$P_2 = \frac{8.5 - 5.5}{2(0.006)} = 250.0000$$

$$P_3 = \frac{8.5 - 5.8}{2(0.009)} = 150.0000$$

$C_t = C_1 + C_2 + C_3 = 6682.5$$/h$
Solving $\lambda$ by the N-R Method: Example 7.4

$$P_D = \sum_{i=1}^{n_g} P_i = f(\lambda) \approx f(\lambda)^{(k)} + \left(\frac{df(\lambda)}{d\lambda}\right)^{(k)} \Delta \lambda^{(k)}$$

$$\Delta \lambda^{(k)} = \frac{\Delta P^{(k)}}{\left(\frac{df(\lambda)}{d\lambda}\right)^{(k)}} = \frac{P_D - f(\lambda)^{(k)}}{\sum_{i=1}^{n_g} \frac{dP_i}{d\lambda}^{(k)}}$$

$$\lambda^{(k+1)} = \lambda^{(k)} + \Delta \lambda^{(k)}$$

until $|\lambda^{(k+1)} - \lambda^{(k)}| \leq \varepsilon$

$$P_i^{(k)} = \frac{\lambda^{(k)} - \beta_i}{2\gamma_i}$$

$$\Delta P^{(k)} = P_D - \sum_{i=1}^{n_g} P_i^{(k)}$$

$$\Delta \lambda^{(k)} = \frac{\Delta P^{(k)}}{\sum \left(\frac{dP_i}{d\lambda}\right)^{(k)}} \sum \frac{1}{2\gamma_i}$$

$\lambda^{(1)} = 6.0$

$$P_1^{(1)} = \frac{6.0 - 5.3}{2(0.004)} = 87.5000$$

$$P_2^{(1)} = \frac{6.0 - 5.5}{2(0.006)} = 41.6667$$

$$P_3^{(1)} = \frac{6.0 - 5.8}{2(0.009)} = 11.1111$$

$$\Delta P^{(1)} = 800 - (87.5 + 41.6667 + 11.1111) = 659.7222$$

$$\Delta \lambda^{(1)} = \frac{659.7222}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}}$$

$$= \frac{659.7222}{263.8888} = 2.5$$

$\lambda^{(2)} = 6.0 + 2.5 = 8.5$

$$P_1^{(2)} = \frac{8.5 - 5.3}{2(0.004)} = 400.0000$$

$$P_2^{(2)} = \frac{8.5 - 5.5}{2(0.006)} = 250.0000$$

$$P_3^{(2)} = \frac{8.5 - 5.8}{2(0.009)} = 150.0000$$

$$\Delta P^{(2)} = 800 - (400 + 250 + 150) = 0.0$$

$$C_t = 6682.5 \ $ / h$$
ED Neglecting Losses but Including Generator Limits

- Considering the maximum (by rating) and minimum (for stability) generation limits,

Minimize \( C_t = \sum_{i=1}^{n_g} C_i = \sum_{i=1}^{n} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \)

Subject to \( \sum_{i=1}^{n_g} P_i = P_D \)

\( P_i(\text{min}) \leq P_i \leq P_i(\text{max}) \quad i = 1, 2, \cdots, n_g \)

\( \iff P_i - P_i(\text{max}) \leq 0, \quad P_i(\text{min}) - P_i \leq 0 \)

\( L = C_t + \lambda (P_D - \sum_{i=1}^{n_g} P_i) + \sum_{i=1}^{n_g} [\mu_i (P_i - P_i(\text{max})) + \gamma_i (P_i(\text{min}) - P_i)] \)

\[ \frac{\partial L}{\partial P_i} = 0 \Rightarrow \frac{\partial C_i}{\partial P_i} - \lambda + \mu_i - \gamma_i = 0 \]

\[ \frac{\partial L}{\partial \lambda} = 0 \Rightarrow \sum_{i=1}^{n_g} P_i = P_D \]

\[ \frac{\partial L}{\partial \mu_i} \leq 0 \quad \mu_i (P_i - P_i(\text{max})) = 0, \quad \mu_i \geq 0 \]

\[ \frac{\partial L}{\partial \gamma_i} \leq 0 \quad \gamma_i (P_i(\text{min}) - P_i) = 0, \quad \gamma_i \geq 0 \]

\( \frac{dC_i}{dP_i} = \lambda \quad (\mu_i = \gamma_i = 0) \)

\( \frac{dC_i}{dP_i} = \lambda - \mu_i \leq \lambda \quad (\gamma_i = 0) \)

\( \frac{dC_i}{dP_i} = \lambda + \gamma_i \geq \lambda \quad (\mu_i = 0) \)

- Except the plants that reach their limits, all plants still operate at equal incremental cost \( \lambda \).

- the plants reaching \( P_i(\text{max}) \) have cost \( \leq \lambda \)

- the plants reaching \( P_i(\text{min}) \) have cost \( \geq \lambda \)
Example 7.6

Consider generator limits (in MW) for Example 7.4, let \( P_D = 975 \text{MW} \)

\[
\begin{align*}
\frac{dC_1}{dP_1} &= 5.3 + 0.008P_1 = \lambda \\
\frac{dC_2}{dP_2} &= 5.5 + 0.012P_2 = \lambda \\
\frac{dC_3}{dP_3} &= 5.8 + 0.018P_3 = \lambda
\end{align*}
\]

Solution:

\[
\begin{align*}
P_1 &= 450 \text{MW} \\
P_2 &= 325 \text{MW} \\
P_3 &= 200 \text{MW} \\
\lambda &= 9.4 \$/\text{MWh} \\
C_t &= 8236.25 \$/\text{h}
\end{align*}
\]
\[
P_i^{(k)} = \frac{\lambda^{(k)} - \beta_i}{2\gamma_i}
\]

\[
\Delta \lambda^{(k)} = \frac{\Delta P^{(k)}}{\sum (\frac{dP_i^{(k)}}{d\lambda})^{(k)}} = \frac{\Delta P^{(k)}}{\sum \frac{1}{2\gamma_i}}
\]

\[
\lambda^{(1)} = 6.0
\]

\[
P_1^{(1)} = \frac{6.0 - 5.3}{2(0.004)} = 87.5000
\]

\[
P_2^{(1)} = \frac{6.0 - 5.5}{2(0.006)} = 41.6667
\]

\[
P_3^{(1)} = \frac{6.0 - 5.8}{2(0.009)} = 11.1111
\]

\[
\Delta P^{(1)} = 975 - (87.5 + 41.6667 + 11.1111) = 834.7222
\]

\[
\Delta \lambda^{(1)} = \frac{834.7222}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{834.7222}{263.8888} = 3.1632
\]

\[
\lambda^{(2)} = 6.0 + 3.1632 = 9.1632
\]

\[
P_1^{(2)} = \frac{9.1632 - 5.3}{2(0.004)} = 482.8947 > P_{1\text{,max}}\), so let } P_1 = 450
\]

\[
P_2^{(2)} = \frac{9.1632 - 5.5}{2(0.006)} = 305.2632
\]

\[
P_3^{(2)} = \frac{9.1632 - 5.8}{2(0.009)} = 186.8421
\]

\[
\Delta P^{(2)} = 975 - (450 + 305.2632 + 186.8421) = 32.8947
\]

\[
\Delta \lambda^{(2)} = \frac{32.8947}{\frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{32.8947}{132.8889} = 0.2368
\]

\[
\lambda^{(3)} = 9.1632 + 0.2368 = 9.4
\]

\[
P_1^{(3)} = 450
\]

\[
P_2^{(3)} = \frac{9.4 - 5.5}{2(0.006)} = 325
\]

\[
P_3^{(3)} = \frac{9.4 - 5.8}{2(0.009)} = 200
\]

\[
\Delta P^{(3)} = 975 - (450 + 325 + 200) = 0.0
\]
Solve the optimal dispatch for $P_D=550$MW

$$C_1 = 500 + 5.3P_1 + 0.004P_1^2 \quad 250 \leq P_1 \leq 450 \quad \text{or} \quad P_1 = 0$$

$$C_2 = 400 + 5.5P_2 + 0.006P_2^2 \quad 150 \leq P_2 \leq 350 \quad \text{or} \quad P_2 = 0$$

$$C_3 = 200 + 5.8P_3 + 0.009P_3^2 \quad 100 \leq P_3 \leq 225 \quad \text{or} \quad P_3 = 0$$

Unit Commitment Problem (mixed-integer optimization)

**Solution 1:**
- $P_1=280$MW
- $P_2=170$MW
- $P_3=100$MW
- $\lambda=7.54$/MWh
- $C_t=4676$$/h$

**Solution 2:**
- $P_1=340$MW
- $P_2=210$MW
- $P_3=0$MW
- $\lambda=8.02$/MWh
- $C_t=4584$$/h$

**Solution 3:**
- $P_1=0$MW
- $P_2=340$MW
- $P_3=210$MW
- $\lambda=9.58$/MWh
- $C_t=47785$$/h$

**Solution 4:**
- $P_1=400$MW
- $P_2=0$MW
- $P_3=150$MW
- $\lambda=8.5$/MWh
- $C_t=4532$$/h$
Transmission Loss

• When transmission distances are very small and load density in the system is very high
  – Transmission losses may be neglected
  – All plants operate at equal incremental production cost to achieve optimal dispatch of generation

• However, in a large interconnected network
  – Power is transmitted over long distances with low load density areas
  – Transmission losses are a major factor affecting the optimal dispatch.