Digital Compensator Design

Difference Equations

SISO, Linear, Causal, Time-Invariant, DT Systems represented by difference equation

\[ y[k] = \sum_{i=1}^{N} a_i y[k - i] + \sum_{i=0}^{M} b_i u[k - i] \]

Output is a linear combination of pas M outputs and N inputs. 
\( y[k>0] \) uniquely determined by difference equation, along with \( M,N \) initial conditions.
Due to linearity,
\[ y[k] = y_f[k] + y_o[k] \]
\( y_f[k] \) is the forced response, particular solution (no ICs)
\( y_o[k] \) is the free response, natural response, or homogeneous solution (no input)
Forced Response

Any discrete time signal can be represented as

\[ u[k] = \sum_{i=0}^{+\infty} u[i] \delta[k - i], \quad k \geq 0 \]

Due to linearity, the forced response is then

\[ y_f[k] = \sum_{i=0}^{k} u[i] h[k - i] = \sum_{i=0}^{k} h[i] u[k - i] \]

which is a discrete convolution summation

Natural Response

The natural response is

\[ y_o[k] = \sum_{i=1}^{N} a_i y_o[k - i] \]

\[ 0 = \sum_{i=0}^{N} a_i y_o[k - i], \quad a_0 = -1 \]

Assume solution takes the form

\[ y_o[k] = Cz^k, \quad z \in \mathbb{C} \]

\[ 0 = C \sum_{i=0}^{N} a_i z^{k-i} = a_0 z^k + a_1 z^{k-1} + \cdots + a_N z^{k-N} \]

\[ 0 = z^N - a_1 z^{N-1} - \cdots - a_N \]
Natural Response

\[ 0 = z^N - a_1 z^{N-1} - \cdots - a_N \]

This is the characteristic equation; it has \( N \) roots when factored.

Most often, the systems we examine will be either real or complex pairs with multiplicity one.

\[ z_i = p_i, \quad i = 1 \ldots N_r \]
\[ z_i = r_i e^{\pm j\theta_i}, \quad i = 1 \ldots N_c \]

with \( N = N_r + 2N_c \). Then, the natural response is some linear combination of these roots

\[ y_o[k] = \sum_{i=1}^{N_r} A_i p_i^k + \sum_{i=1}^{N_c} (B_i r_i^k \cos(k\theta_i) + \bar{B}_i r_i^k \sin(k\theta_i)) \]

If \( M > N \), there will also be a sequence of constants dictated by the initial conditions.

DT Natural Response

Figure 3.11 Time-domain behavior of a single-real pole causal signal as a function of the location of the pole with respect to the unit circle.

Figure 3.13 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.
Z-transform

\[ Z\{h\} \triangleq \sum_{k=0}^{+\infty} h[k]z^{-k} = H(z) \]

Properties:
1. Linearity
2. Delay: \( z^{-1} \) represents a one-sample delay
3. Discrete convolution of two signals is multiplication of their z-transforms

Transfer Function

The forced response (zero ICs)

\[ y_f[k] = \sum_{i=0}^{k} u[i]h[k - i] \]

is, in the z-domain

\[ y_f(z) = H(z)u(z) \]

which has z-domain transfer function

\[ \frac{y_f(z)}{u(z)} = H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 - \sum_{i=1}^{N} a_i z^{-i}} \]

The frequency response of any stable system is equal to \( H(z) \) evaluated with \( z = e^{j\theta}, \theta = \omega T_s \)
Pole Mapping Between Planes

Noteworthy Z-transforms

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<th>Function</th>
<th>Difference Equation</th>
<th>Z-domain TF</th>
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<tr>
<td>delay</td>
<td>$y[k] = u[k - N]$</td>
<td>$H(z) = z^{-N}$</td>
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<tr>
<td>Integrator</td>
<td>$y[k] = y[k - 1] + K_i u[k]$</td>
<td>$H(z) = \frac{K_i}{1 - z^{-1}}$</td>
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<tr>
<td>Differentiator</td>
<td>$y[k] = K_d (u[k] - u[k - 1])$</td>
<td>$H(z) = K_d (1 - z^{-1})$</td>
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</table>
DT Integrator
Magnitude comparison

Example: $f_0 = f_s/10$

Mapping Between Domains

$$\omega' = \frac{2}{T_s} \tan\left(\omega \frac{T_s}{2}\right).$$

$$z(p) = \frac{1 + p \frac{T_s}{2}}{1 - p \frac{T_s}{2}}$$

$$p(z) = \frac{2 (1 - z^{-1})}{T_s (1 + z^{-1})}$$
Compensator Design Approach