The Averaged System

This equation is now the model of a new, equivalent linear system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

where

\[ A = DA_1 + D'A_2 \]
\[ B = DB_1 + D'B_2 \]

which has averaged behavior over one switching period.

This approximation is *perhaps* valid, if
- State waveforms are dominantly linear
- Dynamics of interest are at \( f_{bw} \ll f_s \)

Buck State Space Averaging

In switch position 1

\[ \dot{x}(t) = A_1 x(t) + B_1 u(t) \]

\[
\frac{d}{dt} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1 & -1/R \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v_g(t)
\]

In switch position 2

\[ \dot{x}(t) = A_2 x(t) + B_2 u(t) \]

\[
\frac{d}{dt} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1 & -1/R \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_g(t)
\]
Buck Averaged Model

So, our average model is

\[
\langle \dot{x}(t) \rangle = (D A_1 + D' A_2) \langle x(t) \rangle + (D B_1 + D' B_2) \langle u(t) \rangle
\]

\[
\langle \dot{x}(t) \rangle = \begin{bmatrix} 0 & -1/L \\ 1 & -1/R C \end{bmatrix} \langle x(t) \rangle + \begin{bmatrix} 0 & -1/L \\ 1 & -1/R C \end{bmatrix} \langle u(t) \rangle + \left( D \begin{bmatrix} 1/L \\ 0 \end{bmatrix} + D' \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) V_g
\]

\[
\langle \dot{x}(t) \rangle = \begin{bmatrix} 0 & -1/L \\ 1 & -1/R C \end{bmatrix} \langle x(t) \rangle + \begin{bmatrix} D/L \\ 0 \end{bmatrix} V_g
\]

\[
\begin{cases}
DV_g - \langle v_c(t) \rangle = L \frac{d\langle i_L(t) \rangle}{dt} \\
\langle i_L(t) \rangle - \frac{\langle v_c(t) \rangle}{R} = C \frac{d\langle v_c(t) \rangle}{dt}
\end{cases}
\]

Averaging: Discussion

\[
\langle x(t) \rangle_{T_s} = \frac{1}{T_s} \int_{T_s/2}^{T_s/2} x(\tau) d\tau
\]

\[
G_{av}(j\omega) = \frac{e^{j\omega T_s/2} - e^{-j\omega T_s/2}}{j\omega T_s} = \frac{\sin(\omega T_s/2)}{\omega T_s/2}
\]

Averaging removes switching frequency ripple and harmonics
Discrete Time Nature of PWM
Discrete Time Nature of PWM

Model a switched system as an averaged, time-invariant system with

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

where

\[
A = DA_1 + D' A_2 \\
B = DB_1 + D' B_2
\]

Model a switched system as a discrete-time system with

\[
x[n + 1] = \Phi x[n] + \Psi u[n]
\]

where

\[
\Phi = (\prod_{i=n_{sw}}^{l} e^{A_i t_i}) \\
\Psi = \sum_{i=1}^{n_{sw}} \left\{ (\prod_{k=n_{sw}}^{l+1} e^{A_k t_k}) A_i^{-1} (e^{A_l t_l} - I) B_i \right\}
\]

Historical Perspective

Robert D. Middlebrook
PhD, Stanford, 1955
CalTech Professor, 1955-1998

Slobodan Cúk
PhD CalTech, 1976
CalTech Prof, 1977-1999

Dennis John Packard
PhD, CalTech 1976

Modelling, analysis, and design of switching converters

Discrete modeling and analysis of switching regulators

Large Signal Modeling of SMPS

Discrete Time Modeling

• Every subcircuit is a passive, linear circuit
• Passive, linear circuits can be solved in closed-form
  – Can model states at discrete times without averaging
• Only assumptions required
  – Independent inputs are DC or slowly varying
Solution to State Space Equation

Closed form solution to state space equation

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

Multiply both sides by \( e^{-At} \)

\[ e^{-At} \dot{x}(t) - e^{-At} Ax(t) = e^{-At} Bu(t) \]

Left-hand side is

\[ \frac{d}{dt} (e^{-At} x(t)) = e^{-At} Bu(t) \]

Solution to State Space Equation

\[ \frac{d}{dt} (e^{-At} x(t)) = e^{-At} Bu(t) \]

Can now be solved by direct integration

\[ e^{-At} x(t) - x(0) = \int_{0}^{t} e^{-At} Bu(\tau) d\tau \]

Rearranging

\[ x(t) = e^{At} x(0) + \int_{0}^{t} e^{-A(t-\tau)} Bu(\tau) d\tau \]
Matrix Exponential

Matrix exponential defined by Taylor series expansion

\[ e^{At} = I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^N}{N!} = \sum_{k=0}^{N} \frac{(At)^k}{k!} \]

Well-known issue with convergence in many cases


Properties of the Matrix Exponential

- Matrix exponential always exists
  - i.e. summation will always converge
- Exponential of any matrix is always invertible, with
  \[ e^A e^{-A} = I \]
First Order Taylor Series Expansion

Linear ripple approximation

\[ e^{At} \approx I + At \]

Valid only if switching frequency much faster than system modes

Simplification for Slow-Varying Inputs

\[ x(t) = e^{At} x(0) + \int_{0}^{t} e^{-A(t-\tau)} Bu(\tau) \, d\tau \]

If \(A\) is invertible and \(u(\tau) \approx U\)

\[ x(t) = e^{At} x(0) + A^{-1}(e^{At} - I)BU \]
Application to Switching Converter
Application to Switching Converter

\[ x(DT_s) = e^{A_1 DT_s} x(0) + A_1^{-1} (e^{A_1 DT_s} - I) B_1 U \]

\[ x(T_s) = e^{A_2 DT_s} x(DT_s) + A_2^{-1} (e^{A_2 DT_s} - I) B_2 U \]
Application to Switching Converter

\[
x(DT_s) = e^{A_1 DT_s} x(0) + A_1^{-1} (e^{A_1 DT_s} - I) B_1 U
\]

\[
x(T_s) = e^{A_2 DT_s} x(DT_s) + A_2^{-1} (e^{A_2 DT_s} - I) B_2 U
\]

\[
\boxed{x(T_s) = e^{A_2 DT_s} e^{A_1 DT_s} x(0) + A_2^{-1} (e^{A_2 DT_s} - I) B_2 U + e^{A_2 DT_s} A_1^{-1} (e^{A_1 DT_s} - I) B_1 U}
\]

General Form

Generally, for \( n_{SW} \) separate switching positions

\[
x(T_s) = \left( \prod_{i=n_{SW}}^{1} e^{A_i t_i} \right) x(0) + \sum_{i=1}^{n_{SW}} \left( \prod_{k=n_{SW}}^{i+1} e^{A_k t_k} \right) A_i^{-1} (e^{A_i t_i} - I) B_i U
\]

Equation is in the form of a discrete-time system with

\[
x[n + 1] = \Phi x[n] + \Psi U[n]
\]

Again, the effect of changing modulation (i.e. \( t_i \)) is hidden in nonlinear terms

\[
\hat{x}[n + 1] = \Phi \hat{x}[n] + \Psi \hat{u}[n] + \Gamma \hat{d}[n]
\]

Find \( \Gamma \) by small-signal modeling
Aside: Comparison to Averaged Modeling

\[ x(T_s) = \left( \prod_{i=n_{sw}}^{1} e^{A_i t_i} \right) x(0) + \sum_{i=1}^{n_{sw}} \left\{ \left( \prod_{k=n_{sw}}^{i+1} e^{A_k t_k} \right) A_i^{-1}(e^{A_i t_i} - I)B_i \right\} U \]

Approximate with straight-line waveforms, \( e^{At} \approx I + At \)

\[ x(T_s) = \left( I + \sum_{i=n_{sw}}^{1} A_i t_i + \cdots \right) x(0) + \sum_{i=1}^{n_{sw}} \left\{ \left( I + \sum_{k=n_{sw}}^{i+1} A_k t_k + \cdots \right) t_i B_i \right\} U \]

Neglect all terms with product of two or more \( t_i \)

\[ x(T_s) = \left( I + \sum_{i=1}^{n_{sw}} A_i t_i \right) x(0) + \sum_{i=1}^{n_{sw}} (t_i B_i) U \]

Continuous time conversion

\[ \dot{x}_{DT}(t) = \frac{x(T_s) - x(0)}{T_s} = \sum_{i=1}^{n_{sw}} \left( A_i t_i \right) x + \sum_{i=1}^{n_{sw}} \left( B_i t_i \right) U \]


Aside: Discrete vs Averaged Modeling

So, averaged and discrete time formulations are equivalent if

- Ripple in states is
  1. Dominantly straight-line, so \( e^{A_i t_i} \approx (I + A_i t_i) \)
  2. Low frequency, such that \( t_i t_j \ll \|A_i A_j\| \)
Steady-State Large-Signal Analysis

\[ x(T_s) = \left( \prod_{i=n_{sw}}^{1} e^{A_i t_i} \right)x(0) + \sum_{i=1}^{n_{sw}} \left\{ \left( \prod_{k=n_{sw}}^{i+1} e^{A_k t_k} \right) A_i^{-1}(e^{A_i t_i} - I)B_i \right\} U \]

In steady-state, \( x(T_s) = x(0) \)

\[ x(T_s) = \left( I - \prod_{i=n_{sw}}^{1} e^{A_i t_i} \right)^{-1} \sum_{i=1}^{n_{sw}} \left\{ \left( \prod_{k=n_{sw}}^{i+1} e^{A_k t_k} \right) A_i^{-1}(e^{A_i t_i} - I)B_i \right\} U \]

Gives explicit solution for steady-state operation of any switching circuit

Small Signal Modeling
Small Signal Modeling

\[
\hat{x}_d = (A_1 x(DT_s) + B_1 U )\hat{d}T_s - (A_2 x(DT_s) + B_2 U )\hat{d}T_s e^{A_2 D'T_s} \hat{x}_d
\]
Complete Small Signal Model

This completes the small-signal model
\[
\dot{x}[n + 1] = \Phi \dot{x}[n] + \Psi \hat{u}[n] + \Gamma \hat{d}[n]
\]
where
\[
\Gamma = e^{A_2 D' T_s} (\left( A_1 - A_2 \right) X_D + \left( B_1 - B_2 \right) U) T_s
\]
with \( X_D = x(DT_s) \) in steady-state

Example Results

* Includes \( \tau_f = 760 \text{ns} \) of delay in feedback loop

Averaged Model
DT Model
Inclusion of Delay

\[ G_{vu}(s) = G_{vu}(s)e^{-st_d} \]

Current Control