The exam consists of 5 questions each of several parts. The exam is closed book. All of the questions can be correctly answered in a reasonable amount of time and space. If you need an excessive amount of time or computations to answer a problem, then you are doing something wrong. Show your work clearly.

1. SET COVERING (20 points): Consider a problem where you wish to identify events that have occurred based on a set of alerts or alarms. You receive two alarms \{a_2, a_3\}. The relationships between events and the alarms are given below. Events e_2 and e_3 occur with equal likelihood and are more than twice as likely to occur as e_1, e_4 and e_5. Formulate an optimization problem to find the minimal set of events that will most likely explain the alarms. Identify clearly all the constraints and the objective function and any other assumptions.

   - Let \( x_i \) be whether an event has occurred.
   - Let \( c_i \) be the likelihood of event \( i \).
   - Then \( c = \{x_1, x_2, x_3, x_4, x_5\}\).
   - Since alarms \( a_2, a_3 \) received let \( b = \{0, 1, 1, 0, 0\}\).
   - The relationship between events and alarms is given by:
     \[
     A = \begin{bmatrix}
     1 & 0 & 0 & 1 & 1 \\
     1 & 0 & 0 & 0 & 0 \\
     0 & 1 & 0 & 0 & 0 \\
     0 & 0 & 1 & 0 & 0 \\
     0 & 0 & 0 & 0 & 0
     \end{bmatrix}
     \]
   - \( \min c^T x \geq A x \geq b \quad x \geq 0 \quad x \in \mathbb{X} \)
   - Binary constraint unneeded.
2. **LINEAR PROGRAM (25 points):** Given the linear programming problem below.

\[
\begin{align*}
\text{min} & \quad 5x_1 + 2x_2 \\
\text{such that} & \quad 3x_1 + x_2 \geq 4 \\
& \quad x_1 - 2x_2 \leq 3.5 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

a) Solve this problem graphically. Identify the solution and **all** the extreme points.

b) Now formulate this problem in standard form and select the extreme point, which is optimal. For this point:
   - identify the basic variables, \(x_B\), non-basic variables, \(x_N\), and the basis \(B\).
3. **INTEGER PROGRAM (10 points):** Given the linear programming problem from problem 2, repeated below, suppose that $x_1$ is now restricted to be integer. Starting with the solution found in question (2a), solve using the branch-and-bound algorithm. (Note the solution at each branch should be obvious and a solution is found quickly for this problem so if you need to do a large number of branches you are making a mistake).

$$\begin{align*}
\text{min } & \quad 5x_1 + 2x_2 \\
\text{such that } & \quad 3x_1 + x_2 \geq 4 \\
& \quad x_1 - 2x_2 \leq 3.5 \\
& \quad x_1, x_2 \geq 0 \quad x_1 \text{ integer}
\end{align*}$$

Solve previous with a branch for $x_1 \leq 1$ and a branch for $x_2 \geq 2$

by inspection

$z = 7$  \quad $x = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$

$z = 10$  \quad $x = \begin{bmatrix} 2 & 0 & 2.5 \end{bmatrix}^T$

Done
4. DUALITY (25 points): Consider the constrained optimization problem below:

\[
\begin{align*}
\min & \quad 2x_1^2 + 3x_2^2 \\
\text{such that} & \quad 3x_1 + 2x_2 \geq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

a) Find the dual of this non-linear program.
b) Repeat if \( x_1 \) and \( x_2 \) are binary. Plot this dual function.

a) Form the Lagrangian:

\[
\mathcal{L}(x, \lambda) = 2x_1^2 + 3x_2^2 + \lambda (4 - 3x_1 - 2x_2)
\]

Minimize over \( x \) by applying \( \frac{\partial \mathcal{L}}{\partial x_1} = 0 \) and \( \frac{\partial \mathcal{L}}{\partial x_2} = 0 \):

\[
\begin{align*}
2 \frac{\partial \mathcal{L}}{\partial x_1} &= 4x_1 + \lambda (-3) = 0 \Rightarrow x_1 = \frac{3}{4} \lambda \\
2 \frac{\partial \mathcal{L}}{\partial x_2} &= 6x_2 + \lambda (-2) = 0 \Rightarrow x_2 = \frac{1}{3} \lambda
\end{align*}
\]

Now substitute:

\[
\omega(\lambda) = 2 (\frac{3}{4} \lambda)^2 + 3 (\frac{1}{3} \lambda)^2 + \lambda (4 - \frac{9}{4} \lambda - \frac{2}{3} \lambda)
\]

\[
= 4 \lambda - \frac{35}{24} \lambda^2, \quad \lambda \geq 0
\]

b) if \( x_1 \) and \( x_2 \) binary, then enumerate:

\[
\begin{align*}
(0, 0) & \Rightarrow \mathcal{L}(x, \lambda) = 4 \lambda \\
(0, 1) & \Rightarrow \mathcal{L}(x, \lambda) = 2 \lambda + 3 \\
(1, 0) & \Rightarrow \mathcal{L}(x, \lambda) = 7 - 2 \lambda \\
(1, 1) & \Rightarrow \mathcal{L}(x, \lambda) = 5 - 2 \lambda
\end{align*}
\]

\[
\omega(\lambda) < \min (4 \lambda, 2 \lambda + 3, 7 - 2 \lambda, 5 - 2 \lambda)
\]

\[
\omega^*(x^*) = 3.5, \quad \lambda^* = 1.5
\]
5. **GRAPH SEARCH (20 points):** Consider the graph below with possible routes to Knoxville leaving from Memphis. The paths are directional as indicated by the arrows. The times shown next to the arcs show the actual driving times. The numbers just below the cities show the estimated remaining time to reach Knoxville (the goal). Then:

a) Formulate a linear program for this problem (but do not solve).

b) Find the shortest path using Dijkstra’s search. Show the steps clearly.

c) Find the shortest path using A* search. Show the steps clearly. Does the estimate of the remaining distance to the goal satisfy the A* requirement for finding an optimal solution?

Note, the numbers here aren’t necessarily accurate values so don’t get hung up on your knowledge of these cities.

![Graph Image]

a) Use labels above to define paths $x_{ij}$ so $X = [X_{12} X_{23} X_{14} X_{36} X_{45} X_{56}]$

then $c^T = [1.5 2.0 3.0 2.5 2.0 2.0 1.5]$

and there are equalities for each of the six nodes $A \mathbf{x} = \mathbf{b}$ with $b^T = [10000 -17]$

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1
\end{bmatrix}
$$

$$
\begin{align*}
\min c^T x \\
x \geq 0
\end{align*}
$$

$A \mathbf{x} = \mathbf{b}$
Dijkstra search build up the table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.5</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1.5</td>
<td>3.5</td>
<td>3</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>1.5</td>
<td>3.5</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1.5</td>
<td>3.5</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1.5</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Path is 1-2-3-6

C) A* algorithm modifies above with estimate

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(0.6)</td>
<td>(1.5, 5.5)</td>
<td>(3, 4.8)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>(0.6)</td>
<td>(1.5, 5.5)</td>
<td>(3, 4.7)</td>
<td>(3.4, 25)</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>(0.6)</td>
<td>(1.5, 5.5)</td>
<td>(3, 4.7)</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>(0.6)</td>
<td>(1.5, 5.5)</td>
<td>(3, 4.7)</td>
</tr>
</tbody>
</table>

- No need to look at node 5 as it cannot be better than solution found.
- Heuristic satisfies A* criteria