Consider the following problem.

(Grimaldi, 8.1.20) “At a 12-week conference in mathematics, Sharon met seven of her friends from college. During the conference she met each friend at lunch 35 times, every pair of them 16 times, every trio eight times, every foursome four times, each set of five twice, and each set of six once, but never all seven at once. If she had lunch every day during the 84 days of the conference, did she ever have lunch alone?”

One might wonder whether it is in fact possible for Sharon to meet with “...each friend at lunch 35 times, every pair of them 16 times, every trio eight times, every foursome four times, each set of five twice, and each set of six once, but never all seven at once ... during the 84 days ...” This homework assignment asks you to produce a lunch schedule demonstrating that it is in fact possible.

Since you need to specify who Sharon lunches with on each day, and since that lunch schedule would be tedious to produce by hand, you are encouraged to use a computer to produce it.

Denote Sharron’s friends by elements of the set \( F = \{0, 1, 2, 3, 4, 5, 6\} \). If, for example, Sharon lunched with friends 0, 1, 2, 3, 4 on the first day, friends 2, 3, 5, 6 on the second day, and friend 2 on the third day, then the lunch schedule might begin as follows

\[
\{0, 1, 2, 3, 4\} \\
\{2, 3, 5, 6\} \\
\{2\}
\]

The above representation is actually unacceptable, because there is a particularly elegant way to denote subsets of \( F \) — subsets are denoted by nonnegative integers — and you are required to use it: subset \( S = \{e_1, \ldots, e_k\} \) is denoted by the integer \( n \) whose binary representation

\[
n = b_6b_5b_4b_3b_2b_1b_0 \text{ base 2}
\]

has all bits 0 except \( b_{e_1} = \cdots = b_{e_k} = 1 \). Therefore

\[
\{0, 1, 2, 3, 4\} \iff 31 = 0011111 \text{ base 2} \\
\{2, 3, 5, 6\} \iff 108 = 1101100 \text{ base 2} \\
\{2\} \iff 4 = 0000100 \text{ base 2}
\]

Moreover, the lunch schedule must list the subsets from largest integer representation to smallest; thus an acceptable form of the lunch schedule above would begin as follows

\[
108 \\
31 \\
4
\]

To get a feel for the size of the (unconstrained) search space, the number of subsets of \( F \) is

\[
2^{|F|} = 2^7 = 128
\]

Hence for each of 84 days there are 128 possible lunch configurations, and thus the number of (unconstrained) lunch schedules is

\[
128^{84} > 10^{177}
\]

It is infeasible to generate and check \( 10^{177} \) lunch schedules to see whether constrains (“...each friend at lunch 35 times, every pair of them 16 times, every trio eight times, every foursome four times, each set of five twice, and each set of six once, but never all seven at once ...”) can be satisfied.
A lunch schedule may be constructed by considering subsets (lunch configurations) from largest size to smallest size.

- There are no subsets of size 7 (Sharon never lunches with all friends at once).
- The number of subsets of size 6 is
  \[ \binom{7}{6} = 7 \]
  and all such subsets — denote them by \( S(6)_1, \ldots, S(6)_7 \) — must occur in the lunch schedule exactly once. At this point the lunch schedule is \( S(6)_1, \ldots, S(6)_7 \).
- Of those subsets above — which are already committed to the lunch schedule — how many subsets of size 5 are already represented?
  Each size 5 subset \( S \) is contained in \( \binom{7-5}{1} = 2 \) subsets of size 6; if \( x \) is one of the \( 7 - 5 \) elements not in \( S \), then \( |S \cup \{x\}| = 6 \).
  Thus the constraint for subsets of size 5 is already satisfied.
- Of those subsets above — which are already committed to the lunch schedule — how many subsets of size 4 are already represented?
  Each size 4 subset \( S \) is contained in \( \binom{7-4}{2} = 3 \) subsets of size 6; if \( x, y \) are two of the \( 7 - 4 \) elements not in \( S \), then \( |S \cup \{x, y\}| = 6 \).
  Thus the constraint for subsets of size 4 is partially satisfied; each such subset — there are \( \binom{7}{6} = 35 \) subsets of size 4, denote them by \( S(4)_1, \ldots, S(4)_{35} \) — already occurs in the lunch schedule 3 times.
  Thus the constraint for subsets of size 4 is completely satisfied by adding \( S(4)_1, \ldots, S(4)_{35} \) to the lunch schedule. At this point the lunch schedule is \( S(6)_1, \ldots, S(6)_7, S(4)_1, \ldots, S(4)_{35} \).
- Of those subsets above — which are already committed to the lunch schedule — how many subsets of size 3 are already represented?
  Continue The Analysis To Complete A Lunch Schedule...