Relations, Partial Orders, Directed Graphs

Let \( A, B \) be sets. A subset \( R \) of the Cartesian product \( A \times B \) is a relation from \( A \) to \( B \). If \( B = A \), then \( R \) is a relation on \( A \). An alternate notation for \((a, b) \in R\) is \( aRb \). Relation \( R \) is called:

- reflexive iff \( \forall x \in A . xRx \)
- symmetric iff \( \forall x, y \in A . xRy \Rightarrow yRx \)
- transitive iff \( \forall x, y, z \in A . xRy \land yRz \Rightarrow xRz \)
- antisymmetric iff \( \forall x, y \in A . xRy \land yRx \Rightarrow x = y \)

Consider \( R = \{(0,1), (1,0), (0,2)\} \) on \( A = \{0,1,2\} \). Because \((0,0) \notin R\), it follows that \( R \) is not reflexive. Since \((0,1) \in R\) and \((1,0) \in R\), either transitivity or antisymmetry would imply \((0,0) \in R\); thus \( R \) is neither transitive nor antisymmetric. Because \((2,0) \notin R\), it follows that \( R \) is not symmetric.

Consider the “equality” relation on \( A = \{0,1,2\} \),

\[
R = \{(a, a) \mid a \in A \}
\]

It is reflexive (\( xRx \) for all \( x \in A \)), symmetric (\( xRy \Rightarrow x = y \Rightarrow yRx \) for all \( x, y \in A \)), transitive (\( xRy \land yRz \Rightarrow x = y = z \Rightarrow xRz \) for all \( x, y, z \in A \)), and antisymmetric (\( xRy \land yRx \Rightarrow x = y \) for all \( x, y \in A \)).

A relation \( R \) on \( A \) is:

- a partial order if it is transitive, reflexive and antisymmetric (example: equality),
- a total order if it is a partial order and all elements are comparable (\( \forall x, y \in A . xRy \lor yRx \)).
- an equivalence relation if it is transitive, reflexive and symmetric (example: equality).

## Homework

1. Consider the less than or equal relation on \( \mathbb{R} \),

\[
\leq = \{(a,b) \mid a \text{ is less than or equal to } b\}
\]

Is \( \leq \) a partial order? Is it a total order? Is it an equivalence relation?

2. Let \( 2^U \) be the set of all subsets of a set \( U \). Consider the subset relation on \( 2^U \),

\[
\subset = \{(A, B) \mid A \text{ is a subset of } B\}
\]

Is \( \subset \) a partial order? Is it a total order? Is it an equivalence relation?

3. Let \( U \) be a set of propositions. Consider the implication relation on \( U \),

\[
\Rightarrow = \{\langle \alpha, \beta \rangle \mid \alpha \text{ implies } \beta\}
\]

Is \( \Rightarrow \) a partial order? Is it a total order? Is it an equivalence relation?
The composition \( S;R \) of relations \( S \) and \( R \) is the relation
\[
S;R = \{(x, z) \mid \exists y . xSy \land yRz\}.
\]

Given any relation \( R \) on \( A \), the reflexive closure
\[
R \cup \{(x, x) \mid x \in A\}
\]
is reflexive, the symmetric closure
\[
R \cup \{(y, x) \mid (x, y) \in R\}
\]
is symmetric, and the transitive closure
\[
\bigcup_{i=1}^{\infty} R^i \quad (\text{where } R^1 = R, \text{ and } R^{i+1} = R^i; R)
\]
is transitive.

**Homework:** Consider \( R = \{(0, 1), (1, 0), (0, 2)\} \) on \( A = \{0, 1, 2, 3\} \). Find the reflexive closure, the symmetric closure, the transitive closure, and the reflexive transitive closure (i.e., the reflexive closure of the transitive closure).

A relation \( R \) on \( A \) is equivalent to a directed graph \( \text{dg}(R) \) whose vertex set is \( A \), and which contains an edge from \( x \) to \( y \) iff \( xRy \). Conversely, corresponding to directed graph \( G \) is the relation \( r(G) \) defined by \( (x, y) \in r(G) \) iff \( G \) contains an edge from \( x \) to \( y \).

A directed acyclic graph (DAG) contains no directed path whose first and last vertices are the same (a directed path is a sequence of vertices in which there is an edge from each vertex in the sequence to its successor).

Given partial order \( P \), the equivalent graph \( \text{dg}(P) \) is not a DAG; since \( P \) is reflexive, every vertex has a self-loop\(^1\). However, the graph \( \text{dag}(P) \) obtained from \( \text{dg}(P) \) by removing self-loops is a DAG.\(^2\) Conversely, the relation \( r(G) \) corresponding to DAG \( G \) is not a partial order; it will not be reflexive (\( G \) has no self-loops) and it might not be transitive. However, the reflexive transitive closure of \( r(G) \), which we will denote by \( p(G) \), is a partial order.

**Homework:** Sketch \( \text{dg}(R) \) for \( R = \{(0, 1), (1, 0), (0, 2)\} \) on \( A = \{0, 1, 2, 3\} \). Let \( G_1, G_2 \) denote the left, right graph below (respectively). What is \( P = p(G_1) \)? Draw \( \text{dag}(P) \). What is \( p(G_2) \) ?

\(^1\)A self-loop is an edge from a vertex to itself.
\(^2\)Extra Credit: prove it.
A dependency graph is a finite directed graph $G$ representing constraints between tasks; there is an edge from $x$ to $y$ if task $x$ must be completed before task $y$ can start. If $G$ is not a DAG, then completing the tasks is impossible, because some task must be completed before it can start!

If $G$ is a DAG, there exists a total order $S$ such that $r(G) \subset S$. Thus $S$ is a task schedule which satisfies all constraints. The schedule is serial because different tasks $x$ and $y$ are comparable; either $xSy$ (complete $x$ before starting $y$) or $ySx$ (complete $y$ before starting $x$). The following algorithm produces a potentially parallel schedule corresponding to the partial order $p(G)$ which satisfies all constraints yet may allow some tasks to proceed at the same time.

\[
i \leftarrow 0
\]
\[
\text{while } G \neq \emptyset \text{ do}
\]
\[
L[i] \leftarrow \text{Set of nodes with in-degree } 0
\]
\[
\text{if } L[i] = \emptyset \text{ then}
\]
\[
\text{error (input graph is not a DAG)}
\]
\[
\text{remove from } G \text{ all nodes in } L[i] \text{ and their incident edges}
\]
\[
i \leftarrow i + 1
\]
\[
\text{return } L \text{ (of size } n \text{; indexing begins with } 0)
\]

The schedule is $L$; complete tasks in $L[i]$ before starting tasks in $L[i+1]$, but tasks in $L[i]$ can proceed in parallel because each $L[i]$ is an antichain; a set whose elements are incomparable (i.e., they are not ordered by precedence constraints). Thus $L$ is a partition of the vertices of $G$ into antichains. To obtain a serial schedule, arbitrarily order tasks within each $L[i]$.

**Homework:** Produce parallel task schedules for dependency graphs $G_1$, $G_2$ (see previous page).

Element $m \in A$ is maximal if $\forall x . mPx \Rightarrow x = m$ (m does not precede any other element), and $m$ is maximum if $\forall x . xPm$ (all other elements precede $m$). Element $m \in A$ is minimal if $\forall x . xPm \Rightarrow x = m$ (no other element precedes $m$), and $m$ is minimum if $\forall x . mPx$ (m precedes all other elements).

**Homework:** Give an example of a partial order $P$ on $A$ which contains neither maximal nor minimal elements (hint: a finite partial order contains both maximal and minimal elements).

Give examples of maximal and minimal elements for the partial orders $p(G_1)$ and $p(G_2)$ (hint: running the algorithm above, every element of $L[0]$ is minimal, every element of $L[n-1]$ is maximal). Are there maximum or minimum elements for $p(G_1)$ or $p(G_2)$?

Let $B \subset A$. Element $l \in A$ is a lower bound of $B$ if $\forall x \in B . lPx$ (l precedes every element in $B$), and $l$ is a greatest lower bound if it is a maximum element of the set of all lower bounds of $B$. Element $u \in A$ is an upper bound of $B$ if $\forall x \in B . xPu$ (every element in $B$ precedes $u$), and $u$ is a least upper bound if it is a minimum element of the set of all upper bounds of $B$. None of the following need exist: greatest lower bound, least upper bound, upper bound, lower bound.

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3Total order $S$ is often referred to as a topological sort.

4Extra Credit: prove that the size of array $L$ (i.e., the number of antichains in the partition) is the length of the longest directed path in $G$.

5Extra Credit: prove that if a greatest lower bound exists for $B$, then it is unique, and if a least upper bound exists for $B$, then it is unique.
**Homework:** Using the partial order $p(G_2)$, give example subsets $B$ for the following.

- $B$ has no lower bound.
- $B$ has no upper bound.
- $B$ has a greatest lower bound.
- $B$ has a least upper bound.
- $B$ has a lower bound, but no greatest lower bound.
- $B$ has an upper bound, but no least upper bound.

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**Equivalence Relations**

Equivalence relations can be viewed as functions in the sense that: to each equivalence relation $E$ there is an associated function $f$, and to each function $f$ there is an associated equivalence relation $E$.

Given $E$ on $A$, define the associated function

$$f_E : A \rightarrow 2^A$$

$$x \mapsto \{y \mid yEx\}$$

**Homework:** The image $f_E(x)$ of $x$ is the *equivalence class of $x$*. Prove that the range of $f_E$ — the set of equivalence classes — partitions $A$ (the union of the equivalence classes is $A$, and distinct equivalence classes have empty intersection).\(^6\)

Conversely, given function $f : A \rightarrow B$, define the associated equivalence relation on $A$

$$E_f = \{(x, y) \mid f(x) = f(y)\}$$

**Extra Credit:** Prove that $E_f$ is an equivalence relation, whose set of equivalence classes is

$$\{f^{-1}(x) \mid x \in B\}$$

Moreover, for every equivalence relation $E$,

$$E = E_{f_E}$$

\(^6\)Hint: this is in the book!