ECE 522
Power Systems Analysis II

3.2 – Small-Signal Stability

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Content

• 3.2.1 Small-signal stability overview and analysis methods
• 3.2.2 Small-signal stability enhancement

References:
  – EPRI Dynamic Tutorial
  – Chapters 12 and 17 of Kundur’s “Power System Stability and Control”
  – Chapter 3 of Anderson’s “Power System Control and Stability”
3.2.1 Small-Signal Stability
Overview and Analysis Methods
Power Oscillations

The power system naturally enters periods of oscillation as it continually adjusts to new operating conditions or experiences other disturbances. Typically, oscillations have a small amplitude and do not last long. When the oscillation amplitude becomes large or the oscillations are sustained, a response is required:

- A system operator may have the opportunity to respond and eliminate harmful oscillations or,
- less desirably, protective relays may activate to trip system elements.

Figure 8-1. Rubber Band – Weight Analogy
Small Signal Stability

Small signal stability (also referred to as small-disturbance stability) is the ability of a power system to maintain synchronism when subjected to small disturbances.

- In this context, a disturbance is considered to be small if the equations that describe the resulting response of the system may be linearized for the purpose of analysis.

- It is convenient to assume that the disturbances causing the changes already disappear and details on the disturbance are unimportant.

- The system is stable only if it returns to its original state, i.e. a stable equilibrium point (SEP). Thus, only the behaviors in a small neighborhood of the SEP are concerned and can be analyzed using the linear control theory.
Classic-Model SMIB System

With all resistances neglected:

\[
\begin{align*}
T_e &= P_e = P_t = P_B = P_{\text{max}} \sin \delta \\
P_{\text{max}} &= \frac{E'E_B}{X_T}
\end{align*}
\]

Linearize swing equations at \( \delta = \delta_0 \):

\[
\Delta T_e \approx \frac{\partial T_e}{\partial \delta} \Delta \delta = K_S \Delta \delta
\]

Synchronizing torque coefficient:

\[
K_S = P_{\text{max}} \cos \delta_0 = \frac{E'E_B}{X_T} \cos \delta_0
\]

Note:
- \( H \) in s
- \( \Delta \omega_r \) in p.u.
- \( \Delta \delta \) in rad.
- \( K_D \) in p.u.
- \( K_S \) in p.u./rad
State-space representation

\[
\frac{d}{dt} \begin{bmatrix} \Delta \delta \\ \Delta \omega_r \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\frac{K_s}{2H} & -\frac{K_D}{2H} \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta \omega_r \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\Delta T_m}{2H}
\]

\[
\Delta \ddot{\delta} + \frac{K_D}{2H} \Delta \dot{\delta} + \frac{K_s \omega_0}{2H} \Delta \delta = \frac{\omega_0 \Delta T_m}{2H}
\]

- Apply Laplace Transform:

\[
\Delta \delta = \frac{1}{s^2 + \frac{K_D}{2H} s + \frac{K_s \omega_0}{2H}} \times \frac{\omega_0 \Delta T_m}{2H}
\]

- Characteristic equation:

\[
s^2 + \frac{K_D}{2H} s + \frac{K_s \omega_0}{2H} = 0
\]

---

**Figure 12.5** Block diagram of a single-machine infinite bus system with classical generator model
A harmonic oscillator

\[ F = M \cdot \ddot{x} = -K \cdot x - D \cdot \dot{x} \]

\[ \frac{2H}{\omega_0} \Delta \ddot{x} = -K_s \Delta \delta - \frac{K_D}{\omega_0} \Delta \dot{\delta} \quad \text{(if } \Delta T_m = 0) \]

\[ s^2 + \frac{D}{M} s + \frac{K}{M} = 0 \]

\[ s^2 + \frac{K_D}{2H} s + \frac{K_s \omega_0}{2H} = 0 \]

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]

\( \zeta \) – Damping ratio

\( \omega_n \) – Natural frequency

It has two conjugate complex roots and its zero-input response is a damped sinusoidal oscillation:

\[ s_1, s_2 = \sigma \pm j \omega = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} \]

\[ x(t) = Ae^{\sigma t} \sin(\omega t + \varphi) \]

\[ = Ae^{-\zeta \omega_n t} \sin(\omega_n t \sqrt{1 - \zeta^2} + \varphi) \]

The time of decaying to \( 1/e = 36.8\% \):

\[ \tau = -1/\sigma = \frac{1}{\zeta \omega_n} \]
Oscillation Frequency and Damping of an SMIB System

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]
\[ s_1, s_2 = \sigma \pm j\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \]
\[ s^2 + \frac{K_D}{2H} s + \frac{K_S\omega_0}{2H} = 0 \]
\[ (K_S = \frac{E'E_B}{X_T} \cos\delta_0 = P_{\text{max}} \cos\delta_0) \]

\[ \omega_n = \sqrt{\sigma^2 + \omega^2} = \sqrt{K_S \frac{\omega_0}{2H}} = \sqrt{\frac{\omega_0 E'E_B \cos\delta_0}{2HX_T}} \]

\[ \zeta = \frac{-\sigma}{\sqrt{\sigma^2 + \omega^2}} = \frac{1}{2} \frac{K_D}{\sqrt{K_S 2H \omega_0}} = K_D \frac{X_T}{8\omega_0 HE'E_B \cos\delta_0} \]
\[ \sigma = -\zeta\omega_n = -\frac{K_D}{4H} \]

• How do \( \omega_n \) and \( \zeta \) change with the following?
  – if \( H \downarrow \) (lower inertia)
  – if \( X_T \downarrow \) (stronger transmission)
  – if \( \delta_0 \downarrow \) (lower loading)

Note the units:
- \( \Delta\omega_r \) is in p.u.
- \( \Delta\delta \) is in rad.
- \( K_D \) is in p.u
- \( K_S \) is in p.u/rad
System Response after a Small Disturbance

\[ \frac{d}{dt} \begin{bmatrix} \Delta \delta \\ \Delta \omega_r \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\omega_0}{2H} \\ -\frac{K_s}{2H} & -\frac{K_D}{2H} \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta \omega_r \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\Delta T_m}{2H} \]

\[ x_1 = \Delta \delta \quad x_2 = \Delta \omega_r = \Delta \dot{\delta} / \omega_0 \]

\[ \Delta u = \frac{\Delta T_m}{2H} \]

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_n^2 / \omega_0 & -2\zeta \omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u \]

\[ \dot{x}(t) = Ax(t) + B\Delta u(t) \]

\[ y(t) = Cx(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

(Assuming the angle and speed to be directly measured)

Apply Laplace transform:

\[ sX(s) - x(0) = AX(s) + B\Delta U(s) \]

\[ \Delta U(s) = \frac{\Delta u}{s} \]

\[ Y(s) = X(s) = (sI - A)^{-1}[x(0) + B\Delta U(s)] \]

Zero-input Zero-state

\[ X(s) = \frac{\begin{bmatrix} s + 2\zeta \omega_n & \omega_0 \\ -\omega_n^2 / \omega_0 & s \end{bmatrix}}{s^2 + 2\zeta \omega_n s + \omega_n^2} \left[ x(0) + B\Delta U(s) \right] \]

Zero-input Zero-state
Zero-input response

• E.g. when the rotor is suddenly perturbed by a small angle \( \Delta \delta (0) \neq 0 \) and assume \( \Delta \omega_r (0) = 0 \)

\[
\Delta \delta (s) = \frac{(s + 2 \zeta \omega_n) \Delta \delta (0)}{s^2 + 2 \zeta \omega_n s + \omega_n^2}
\]

\[
\Delta \omega_r (s) = -\frac{\omega_n^2 \Delta \delta (0) / \omega_0}{s^2 + 2 \zeta \omega_n s + \omega_n^2}
\]

Inverse Laplace transform

\[\Delta \delta \text{ in rad} = \frac{\Delta \delta (0)}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega t + \theta)\]

\[\Delta \omega_r \text{ in rad/s} = -\frac{\omega_n \Delta \delta (0)}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega t\]

Damping angle: \( \theta = \cos^{-1} \zeta \)

Zero-state response

• E.g. when there is a small increase in mechanical torque \( \Delta T_m (= \Delta P_m \text{ in pu}) \)

\[
\Delta \delta (s) = \frac{\omega_0 \Delta u}{s (s^2 + 2 \zeta \omega_n s + \omega_n^2)}
\]

\[
\Delta \omega_r (s) = \frac{\Delta u}{s^2 + 2 \zeta \omega_n s + \omega_n^2}
\]

\[\Delta \delta \text{ in rad} = \frac{\omega_0 \Delta T_m}{2H \omega_n^2} \left[ 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin (\omega t + \theta) \right] \]

\[\Delta \omega_r \text{ in rad/s} = \frac{\omega_0 \Delta T_m}{2H \omega_n \sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega t\]
Example

- Exp. 11.2 & 11.3 in Saadat’s book
- \( H=9.94\text{s}, K_D=0.138\text{pu}, T_m=0.6 \text{ pu with PF}=0.8. \)
Find the responses of the rotor angle and frequency under these disturbances
  
1. \( \Delta\delta(0)=10^\circ=0.1745 \text{ rad} \)
2. \( \Delta P_e=0.2\text{pu} \)

**Zero-input response:** \( \Delta\delta(0)=10^\circ \)

**Zero-state response:** \( \Delta P_e=0.2\text{pu} \)

\[ \delta(0)=16.79+10=26.79^\circ \]

\[ \delta(\infty)=16.79+5.76=22.55^\circ \]
<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inter-area or intra-area modes</strong> (0.1-0.7Hz): machines in one part of the system swing against machines in other parts</td>
<td><strong>Inter-area model</strong> (0.1-0.3Hz): involving all the generators in the system; the system is essentially split into two parts, with generators in one part swinging against machines in the other parts. <strong>Intra-area mode</strong> (0.4-0.7Hz): involving subgroups of generators swinging against each other.</td>
</tr>
<tr>
<td><strong>Local modes</strong> (0.7-2Hz): oscillations involve a small part of the system</td>
<td><strong>Local plant modes</strong>: associated with rotor angle oscillations of a single generator or a single plant against the rest of the system; similar to the single-machine-infinite bus system.</td>
</tr>
<tr>
<td><strong>Inter-machine or interplant modes</strong></td>
<td><strong>Inter-machine or interplant modes</strong>: associated with oscillations between the rotors of a few generators close to each other.</td>
</tr>
<tr>
<td><strong>Control or torsional modes</strong> (2Hz – )</td>
<td>Due to inadequate tuning of the control systems, e.g. generator excitation systems, HVDC converters and SVCs, or torsional interaction (sub-synchronous resonance) with power system control</td>
</tr>
</tbody>
</table>
High & Low Frequency Oscillations

• Whenever power flows, $I^2R$ losses occur. These energy losses help to reduce the amplitude of the oscillation.

• High frequency (>1.0 HZ) oscillations are damped more rapidly than low frequency (<1.0 HZ) oscillations. The higher the frequency of the oscillation, the faster it is damped.

• Power system operators do not want any oscillations. However, when oscillations occur, it is better to have high frequency oscillations than low frequency oscillations.

• The power system can naturally dampen high frequency oscillations. Low frequency oscillations are more damaging to the power system, which may exist for a long time, become sustained (undamped) oscillations, and even trigger protective relays to trip elements.
Blackout on August 10, 1996

1. Initial event (15:42:03):
   Short circuit due to tree contact →
   Outages of 6 transformers and lines

2. Vulnerable conditions (minutes)
   Low-damped inter-area oscillations →
   Outages of generators and tie-lines

3. Blackouts (seconds)
   Unintentional separation →
   Loss of 24% load

Diagram:
- Malin-Round Mountain #1 MW
- Time in Seconds
- Transient instability (blackouts)
Oscillation Modes of a Multi-machine System in the Classic Model

\[ \frac{2H_i}{\omega_0} \frac{d^2 \delta_i}{dt^2} = P_{mi} - P_{ei} \quad i = 1, 2, \ldots, n \]  

(Ignoring damping)

\[ P_{ei} = E_i^2 G_{ii} + \sum_{j=1, j\neq i}^{n} P_{ij} = E_i^2 G_{ii} + \sum_{j=1, j\neq i}^{n} E_i E_j Y_{ij} \cos(\delta_{ij} - \delta_i) = E_i^2 G_{ii} + \sum_{j=1, j\neq i}^{n} E_i E_j \left( B_{ij} \sin \delta_{ij} + G_{ij} \cos \delta_{ij} \right) \]

\[ \delta_{ij} = \delta_i - \delta_j, \quad B_{ij} = Y_{ij} \sin \theta_{ij}, \quad G_{ij} = Y_{ij} \cos \theta_{ij} \]

Linearization at \( \delta_{ij0} \):

\[ \delta_{ij} = \delta_{ij0} + \delta_{ij\Delta} \]

\[ \sin \delta_{ij} \approx \sin \delta_{ij0} + \delta_{ij\Delta} \cos \delta_{ij0} \]

\[ \cos \delta_{ij} \approx \cos \delta_{ij0} - \delta_{ij\Delta} \sin \delta_{ij0} \]

\[ \frac{2H_i}{\omega_0} \frac{d^2 \delta_{ij\Delta}}{dt^2} + \sum_{j=1, j\neq i}^{n} K_{sij} \delta_{ij\Delta} = 0 \quad i = 1, 2, \ldots, n \]

\[ \frac{2H_i}{\omega_0} \frac{d^2 \delta_{ij\Delta}}{dt^2} + \sum_{j=1, j\neq i}^{n} E_i E_j \left( B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right) \delta_{ij\Delta} = 0 \]

Synchronizing power coefficient

\[ K_{sij} = \left. \frac{\partial P_{ij}}{\partial \delta_{ij}} \right|_{\delta_{ij0}} = E_i E_j \left( B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right) \]

compared to \( K_s = \frac{E' E_B}{X_T} \cos \delta_0 \)
\[ \frac{2H_i}{\omega_0} \frac{d^2 \delta_{i\Delta}}{dt^2} + \sum_{j=1}^{n} K_{sij} \delta_{ij\Delta} = 0 \quad i = 1, 2, \ldots, n \]

Note: There are only \((n-1)\) independent equations because \(\sum \delta_{ij} = 0\), so we need to formulate the \((n-1)\) independent relative rotor angle equations with one reference machine, e.g., the \(n\)-th machine.

\[
\frac{d^2 \delta_{i\Delta}}{dt^2} - \frac{d^2 \delta_{n\Delta}}{dt^2} + \frac{\omega_0}{2H_i} \sum_{j=1, j \neq i}^{n} K_{sij} \delta_{ij\Delta} - \frac{\omega_R}{2H_n} \sum_{j=1}^{n-1} K_{snj} \delta_{nj\Delta} = 0, \quad i = 1, \ldots, n-1
\]

Consider each \(\delta_{in\Delta} = \delta_{i\Delta} - \delta_{n\Delta}\)

\[
\frac{d^2 \delta_{in\Delta}}{dt^2} + \left( \frac{\omega_0}{2H_i} \sum_{j=1, j \neq i}^{n} K_{sij} + \frac{\omega_0}{2H_n} K_{sni} \right) \delta_{in\Delta} + \sum_{j=1, j \neq i}^{n-1} \left( \frac{\omega_0}{2H_n} K_{snj} - \frac{\omega_0}{2H_i} K_{sij} \right) \delta_{jn\Delta} = 0, \quad i = 1, \ldots, n-1
\]

\[
\frac{d^2 \delta_{in\Delta}}{dt^2} + \sum_{j=1}^{n-1} \alpha_{ij} \delta_{jn\Delta} = 0 \quad i = 1, 2, \ldots, n-1
\]

\[
\alpha_{ii} = \frac{\omega_0}{2H_i} \sum_{j=1, j \neq i}^{n} K_{sij} + \frac{\omega_0}{2H_n} K_{sni}, \quad \alpha_{ij} = \frac{\omega_0}{2H_n} K_{snj} - \frac{\omega_0}{2H_i} K_{sij}
\]
State-space representation

Let $x_1, x_2, \cdots, x_{n-1} = \delta_{1n\Delta}, \delta_{2n\Delta}, \cdots, \delta_{(n-1)n\Delta}$ and

\[
x_n, x_{n+1}, \cdots, x_{2n-2} = \dot{\delta}_{1n\Delta}, \dot{\delta}_{2n\Delta}, \cdots, \dot{\delta}_{(n-1)n\Delta}
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n \\
\vdots \\
\dot{x}_{n+1} \\
\vdots \\
\dot{x}_{2n-2}
\end{bmatrix} =
\begin{bmatrix}
0 \\
-\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1(n-1)} \\
-\alpha_{21} & -\alpha_{22} & \cdots & -\alpha_{2(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{(n-1)1} & -\alpha_{(n-1)2} & \cdots & -\alpha_{(n-1)(n-1)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n \\
\vdots \\
x_{n+1} \\
\vdots \\
x_{2n-2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n \\
\vdots \\
\dot{x}_{n+1} \\
\vdots \\
\dot{x}_{2n-2}
\end{bmatrix} =
\begin{bmatrix}
0 & -1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n \\
\vdots \\
x_{n+1} \\
\vdots \\
x_{2n-2}
\end{bmatrix}
\]

• Its characteristic equation $|\lambda^2 I - A| = 0$ has $2(n-1)$ imaginary roots, which occur in $(n-1)$ complex conjugate pairs

• An $n$-machine system has $(n-1)$ modes

Read Anderson’s Examples 3.2 and 3.3 about the linearization and eigen-analysis on the IEEE 9-bus system
Formulation of General Multi-machine State Equations

- The linearized model of each dynamic device:

\[
\begin{align*}
\dot{x}_i &= A_i x_i + B_i \Delta v \\
\Delta i_i &= C_i x_i - Y_i \Delta v
\end{align*}
\]

\(x_i\) – Perturbed values of state variables
\(i_i\) – Current injection into network from the device
\(\Delta v\) – Vector of network bus voltages

- \(B_i\) and \(Y_i\) have non-zero element corresponding only to the terminal voltage of the device and any remote bus voltages used to control the device

- \(\Delta i_i\) and \(\Delta v\) both have real and imaginary components

- Such state equations for all the dynamic devices in the system may be combined into the form:

\[
\begin{align*}
\dot{x} &= A_D x + B_D \Delta v \\
\Delta i &= C_D x - Y_D \Delta v
\end{align*}
\]

\(x\) is the vector of state variables of the complete system
\(A_D\) and \(C_D\) are block diagonal matrices composed of \(A_i\) and \(C_i\) associated with the individual devices

- Node equation of the transmission network:

\[
\Delta i = Y_N \Delta v
\]

- The overall system state equation:

\[
\begin{align*}
\dot{x} &= A_D x + B_D (Y_N + Y_D)^{-1} C_D x = Ax \\
A &= A_D + B_D (Y_N + Y_D)^{-1} C_D
\end{align*}
\]

- Read Kundur’s sec. 12.7 for other related information, e.g. load model linearization and selection of a reference rotor angle
Modal analysis (eigen-analysis) on an n-dimensional nonlinear system

A power system can be described by

\[ \dot{x} = f(x, u) \]
\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \]

Linearization at the equilibrium \( x_0 \): consider a perturbation at \( x_0 \) and \( u_0 \)

\[ \dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r) \quad i = 1, 2, \ldots, n \]

Equilibrium \( x_0 \) (with \( u_0 \)):

\[ \dot{x}_0 = f(x_0, u_0) = 0 \]
\[ x = x_0 + \Delta x \]
\[ u = u_0 + \Delta u \]

\[ \dot{x} = \dot{x}_0 + \Delta \dot{x} = f((x_0 + \Delta x), (u_0 + \Delta u)) \]

Linearization at the equilibrium \( x_0 \): consider a perturbation at \( x_0 \) and \( u_0 \)

\[ \Delta \dot{x} = A \Delta x + B \Delta u \]

\( A \) is the Jacobian matrix of \( f \)

\[ A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_r} \end{bmatrix} \]

\[ s \Delta X(s) - \Delta X(0) = A \Delta X(s) + B \Delta U(s) \]

\[ \Delta X(s) = (sI - A)^{-1} \left[ \Delta X(0) + B \Delta U(s) \right] \]

\[ = \frac{\text{adj}(sI - A)}{\det(sI - A)} [\Delta X(0) + B \Delta U(s)] \]

Characteristic equation of \( A \)

\[ \det(A - \lambda I) = 0 \]

Poles of \( \Delta X(s) \)

\[ \rightarrow \] Eigenvalues of \( A \), i.e. \( \lambda = \lambda_1 \cdots \lambda_n \)
Eigen-vectors

For any $\lambda_i$, the column vector $\phi_i$ satisfying $A\phi_i = \lambda_i \phi_i$ is called a right eigenvector of $A$ associated with $\lambda_i$

Modal matrix $\Phi = [\phi_1, \phi_2, \cdots, \phi_n]$

$A\Phi = \Phi\Lambda \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$\Phi^{-1}A\Phi = \Lambda$ if $A$ has distinct eigenvalues (usually true for a real-world system)

Similarly the row vector $\psi_j$

$\psi_jA = \lambda_j \psi_j \quad \rightarrow$ Left eigenvector associated with $\lambda_j$

$\Psi = \left[ \psi_1^T, \psi_2^T, \cdots, \psi_n^T \right]^T$

The left and right eigenvectors corresponding to different eigenvalues are orthogonal (row vectors of $\Phi^{-1}$ are left eigenvectors of $A$, so we may let $\Psi = C \Phi^{-1}$ where $C$ is a diagonal matrix or simply equal to $I$ if normalized)

$\Psi\Phi = I \quad \Leftrightarrow \quad \psi_j \phi_i = 0$ if $i \neq j$, or $\psi_i \phi_i = 1$
Free (zero-input) response and stability

\[ \Delta \dot{x} = A \Delta x \quad \rightarrow \quad \text{Linearized system without external forcing} \]

- To eliminate the cross-coupling between the state variables, consider a new state vector \( z \)

\[ \dot{z}_i = \lambda_i z_i \rightarrow z_i(t) = z_i(0) e^{\lambda_i t} \]

\[ \dot{z} = \Lambda z = \Phi^{-1} A \Phi z \quad \Leftrightarrow \quad \Phi \dot{z} = A \Phi z \quad \Leftrightarrow \quad \Delta x(t) = \Phi z(t) = [\phi_1 \phi_2 \cdots \phi_n] \]

\[ z(t) = \Phi^{-1} \Delta x(t) = \Psi \Delta x(t) \]

\[ z_i(0) = \psi_i \Delta x(0) = c_i \]

\[ \Delta x(t) = \sum_{i=1}^{n} \phi_i \psi_i \Delta x(0) e^{\lambda_i t} = \sum_{i=1}^{n} \phi_i c_i e^{\lambda_i t} \]

\[ \Delta x_k(t) = \sum_{i=1}^{n} \phi_{ki} c_i e^{\lambda_i t} = \phi_{k1} c_1 e^{\lambda_1 t} + \ldots + \phi_{kn} c_n e^{\lambda_n t} \]

Free response is a linear combination of \( n \) modes

- Each eigenvalue \( \lambda = \sigma \pm j \omega \)
  - A real eigenvalue (\( \omega = 0 \)) corresponds to a non-oscillatory mode.
    - a decaying mode has \( \sigma < 0 \); a mode with \( \sigma > 0 \) has aperiodic instability.
  - Complex eigenvalues (\( \omega \neq 0 \)) occur in conjugate pairs; each pair corresponds to one oscillatory mode
    - Frequency of oscillation in Hz: \( f = \omega / 2\pi \)
    - Damping ratio (rate of decay) of the oscillation amplitude

\[ \xi = \frac{-\sigma}{\sqrt{\sigma^2 + \omega^2}} \]
Mode Shape and Mode Composition

\[ \Delta x(t) = \Phi z(t) = [\phi_1 \phi_2 \cdots \phi_n][z_1(t), z_2(t), \ldots, z_n(t)]^T \]
\[ \Delta x_k(t) = \sum_{i=1}^{n} \phi_{ki} z_i(t) \]
\[ z(t) = \Psi \Delta x(t) = [\psi_1^T, \psi_2^T, \ldots, \psi_n^T]^T [\Delta x_1(t), \Delta x_2(t), \ldots, \Delta x_n(t)]^T \]
\[ z_i(t) = \sum_{k=1}^{n} \psi_{ik} \Delta x_k(t) \]

- The variables \( \Delta x_1, \Delta x_2, \ldots, \Delta x_n \) are original state variables chosen to represent the dynamic performance of the system.
- Variables \( z_1, z_2, \ldots, z_n \) are transformed state variables; each is associated with only one mode. In other words, they are directly related to the modes.
- The right eigenvector \( \phi_i \) gives the mode shape of the \( i \)th mode, i.e. the relative activity of the original state variables when the \( i \)th mode is excited: The \( k \)th element of \( \phi_i \), i.e. \( \phi_{ki} \), measures the activity of state variable \( x_k \) in the \( i \)th mode.
- The left eigenvector \( \psi_i \) gives the mode composition of the \( i \)th mode, i.e. what weighted composition of original state variables is needed to construct the mode: The \( k \)th element of \( \psi_i \), i.e. \( \psi_{ik} \), weights the contribution of \( x_k \)’s activity to the \( i \)th mode.
Participation factor

\[
\Psi \Phi = I \rightarrow (\Psi \Phi)_{ii} = \sum_{k=1}^{n} \psi_{ik} \phi_{ki} \triangleq \sum_{k=1}^{n} p_{ik} = 1
\]

- \( \phi_{ki} \) measures the activity of \( x_k \) in the \( i^{th} \) mode
- \( \psi_{ik} \) weights the contribution of this activity to the mode
- **Participation factor** \( p_{ki} = \psi_{ik} \phi_{ki} \) measures the participation of the \( k^{th} \) state variable \( x_k \) in the \( i^{th} \) mode.
- \( p_{ki} \) is dimensionless and hence invariant under changes of scale on the variables
- **Question:** considering \( \Psi = \Phi^{-1} \), why do we have to define the mode shape, mode composition, and participation factors, separately?

Learn Kundur’s Example 12.2 on a SMIB system