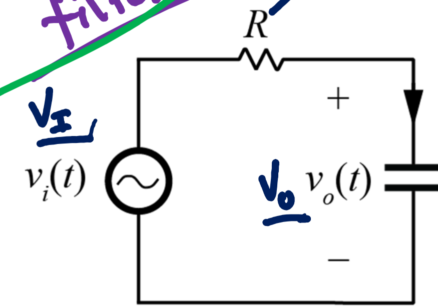
Frequency Response

Low-pass filter



$$V_o = V_I \frac{Z_C}{Z_R + Z_C} = V_I \frac{-j/\omega C}{R - j/\omega C}$$

$$V_o = V_I \underbrace{\frac{1}{jR\omega C + 1}}_{\text{Frequency Response}}$$

$$\frac{V_o}{V_I} = \frac{1}{j\omega RC + 1} = H(j\omega)$$

= Frequency Response → some complex number that varies with ω and describes the I-O relationship of the circuit or system.

$$H(j\omega) = |H(j\omega)| \measuredangle \angle H(j\omega)$$

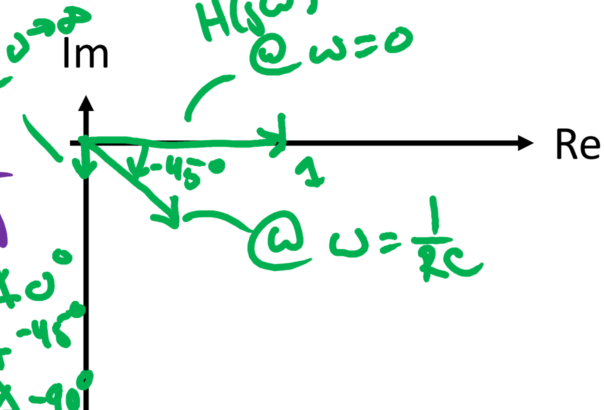
$$V_o = V_I \cdot H(j\omega)$$
$$= |V_I| \cdot |H(j\omega)| \measuredangle [\angle V_I + \angle H(j\omega)]$$

$$H(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} \measuredangle \tan^{-1}(-\omega RC)$$

@ $\omega \to 0$ ⟶ $H(j\omega) = 1 \measuredangle 0°$

@ $\omega = \frac{1}{RC}$ ⟶ $H(j\omega) = \frac{1}{\sqrt{2}} \measuredangle -45°$

@ $\omega \to \infty$ ⟶ $H(j\omega) = 0 \measuredangle -90°$



Frequency Response

example

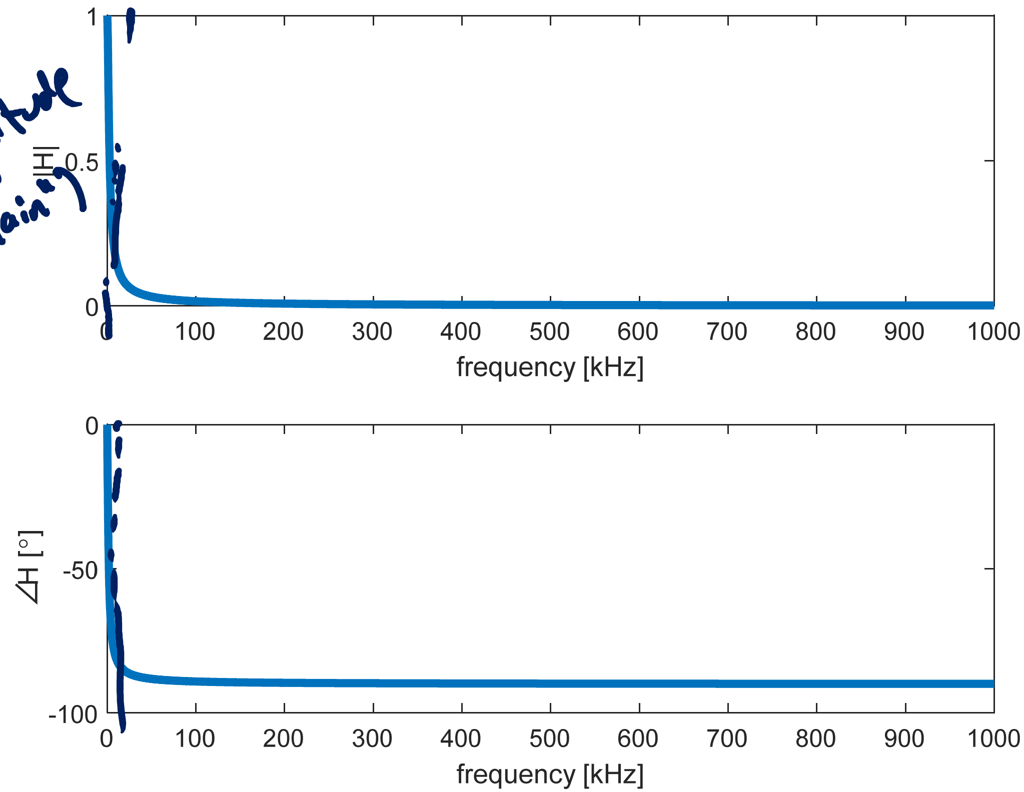
R = 10

C = 10nF

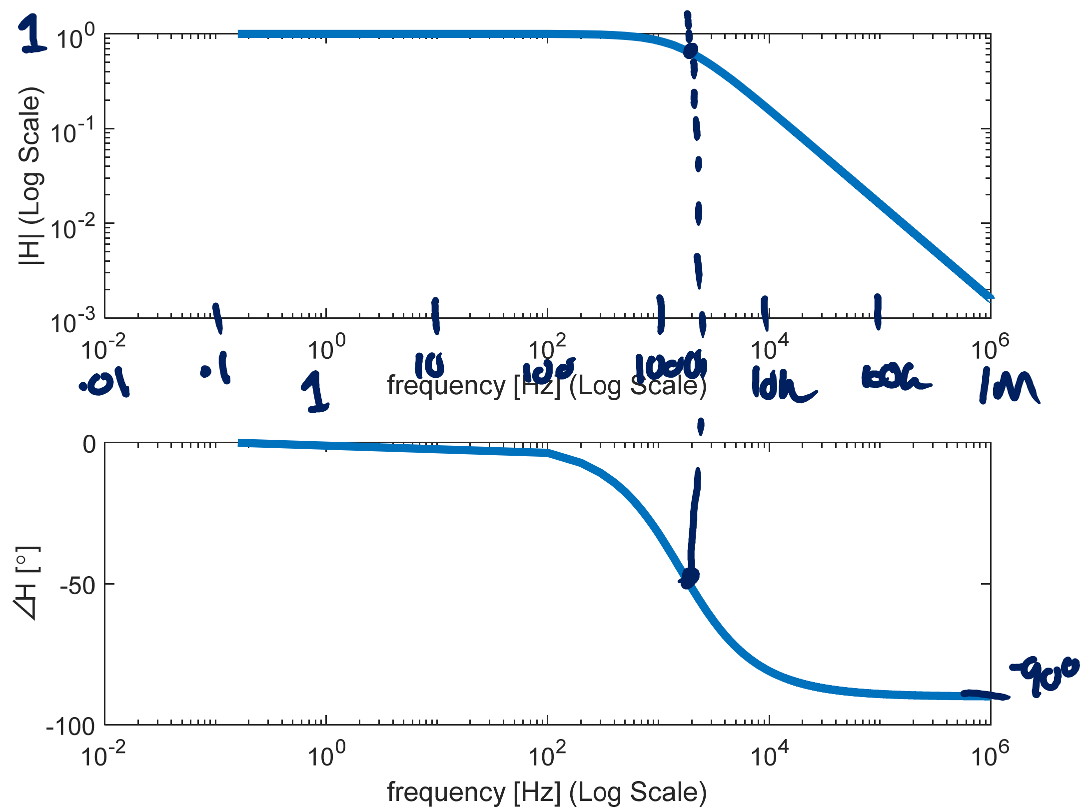
$\frac{1}{RC}$ = 10 krad/sec = 1.6kHz

Magnitude ("Gain")

phase



Bode Plot – Frequency Response



Fourier Series

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

constant   @ wo & above

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if

1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

for a_0 (constant / DC term)

$$\boxed{a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\, dt}$$

for a_k: find average value of $f(t) \cdot \cos(n\omega_0 t)$

$$\frac{1}{T_0}\int_0^{T_0} f(t) \cdot \cos(n\omega_0 t)$$

Assuming fourier series is valid, this is

$$\frac{1}{T_0}\int_0^{T_0} \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right] \cos(n\omega_0 t)$$

$$= \frac{1}{T_0}\int_0^{T_0} \cancel{a_0 \cos(n\omega_0 t)\,dt} \xrightarrow{} \emptyset \quad + \frac{1}{T_0}\int_0^{T_0}\sum_{k=1}^{\infty} a_k \cos(k\omega_0 t)\cos(n\omega_0 t)\,dt$$

$n \in \mathbb{Z}^+$

$$+ \frac{1}{T_0}\int_0^{T}\sum_{k=1}^{\infty} b_k \underbrace{\sin(k\omega_0 t)}_{\cos(k\omega_0 t - 90^\circ)}\cos(n\omega_0 t)\,dt$$

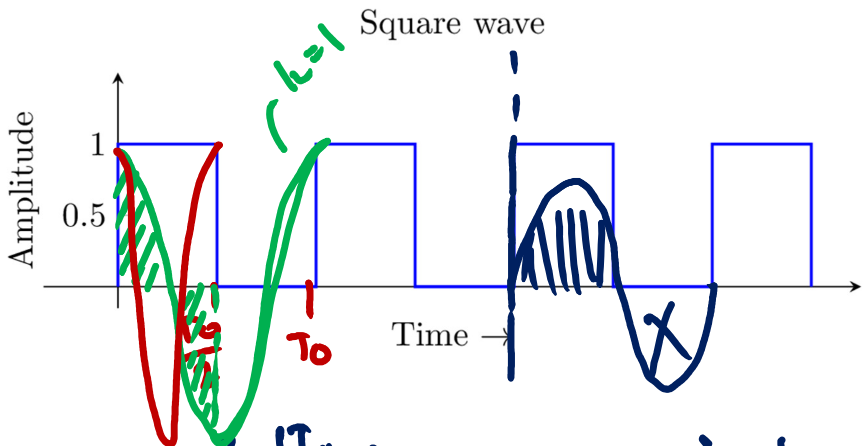
$$= \frac{1}{T_0}\int_0^{T_0}\sum_{k=1}^{\infty} a_k \frac{1}{2}\left(\cancel{\cos((k+n)\omega_0 t)} + \underline{\cos((k-n)\omega_0 t)}\right)dt \quad \emptyset$$

$$+ \frac{1}{T_0}\int_0^{T_0}\sum_{k=1}^{\infty} b_k \frac{1}{2}\left(\cancel{\cos((k+n)\omega_0 t - 90^\circ)} + \cancel{\cos((k-n)\omega_0 t - 90^\circ)}\right)dt \quad \emptyset \quad \emptyset$$

$$\frac{1}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t) = \begin{cases} a_k/2, & \text{if } k=n \\ \emptyset, & \text{otherwise} \end{cases}$$

$$\boxed{a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega_0 t)} \qquad \boxed{b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega_0 t)}$$

Example Calculation



$$\omega_0 = \frac{2\pi}{T_0}$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt = \frac{1}{2}$$

$$= \frac{1}{T_0}\int_0^{T_0/2} 1\cdot dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} \emptyset\, dt$$

$$a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega_0 t)\,dt = \emptyset$$

for all k

$$b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{T_0/2} (1)\sin(k\omega_0 t)\,dt$$