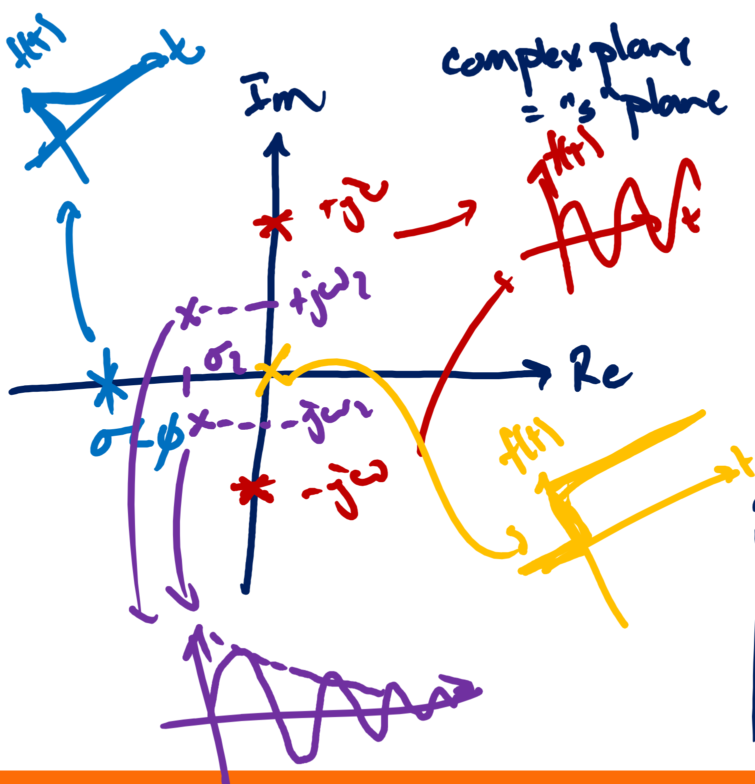
Complex Frequency

$F(s) = \mathcal{L}\{f(t)\}$

$s = \sigma + j\omega$ is a "complex frequency"

Laplace makes our signal f(t) look like a superposition of terms $K_i e^{st} = K_i e^{(\sigma + j\omega)t} = K_i e^{\sigma t} e^{j\omega t}$



if $\sigma = 0$, $s = 0 + j\omega \rightarrow$ sinusoids (w/ $-j\omega$ included for Euler) $A\cos(\omega t + \phi)$

$\omega = 0$, $s = \sigma + j0 \rightarrow$ exponentials $Be^{\sigma t}$
converging if $\sigma < 0$

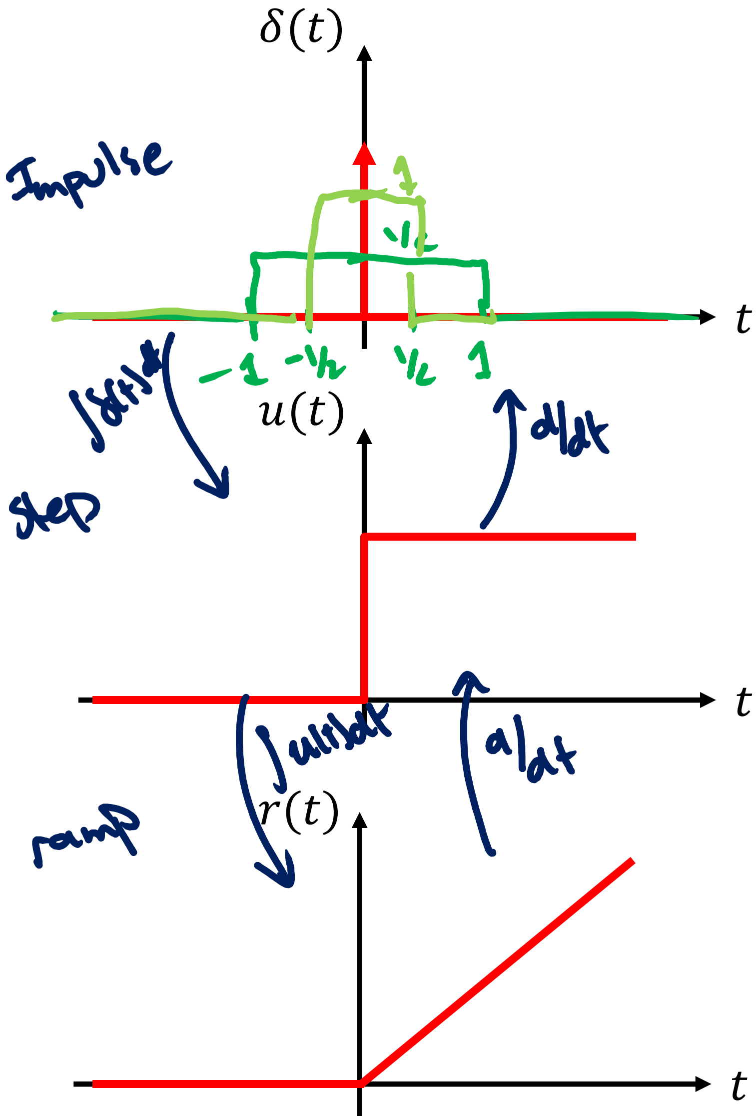
if $\omega = 0$ & $\sigma = 0$, $s = 0 \rightarrow$ constants C

nothing zero, $s = \sigma + j\omega \rightarrow$ exponentials * sinusoids $De^{\sigma t}\cos(\omega t + \phi)$

Conditions for F(s) to exist

1. f(t) must be integrable over any finite time range
2. $\lim_{t\to\infty} e^{-\sigma_0 t}|f(t)|dt$ exists for some real σ_0

Impulse, Step, and Ramp Functions



$$\delta(t)\begin{cases}0 & t \neq 0\\ \infty & t = 0\end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$$u(t)\begin{cases}0 & t < 0\\ 1 & t > 0\end{cases}$$

$$r(t) = tu(t) = \begin{cases}0 & t \leq 0\\ t & t \geq 0\end{cases}$$

Sifting Property of Impulse Function $\delta(t)$

Because $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t)\, dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-t_0)\, dt = f(t_0)$$

Example Signal Laplace Transforms

$f(t) = u(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} e^{-st} u(t)\, dt = \int_{0}^{\infty} e^{-st}\, dt$$

$$= \left[\frac{-1}{s} e^{-st}\right]\Bigg|_{t\to 0}^{t\to\infty} = \left[0 - \left(\frac{-1}{s}\right)\right] = \frac{1}{s}$$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}, \text{ if } \mathrm{Re}\{s\} > 0}$$

Region of Convergence (R.O.C)

$f(t) = e^{-at} u(t)$

$a \in \mathbb{R}^+$

$$\mathcal{L}[f(t)] = \int_{0}^{\infty} e^{-st} e^{-at} u(t)\, dt = \int_{0^-}^{\infty} e^{-(s+a)t}\, dt$$

using previous

$$\boxed{\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \text{ if } \mathrm{Re}\{s+a\} > 0}$$

Generalize: $\mathcal{L}\{f(t) e^{-at}\} = F(s+a)$ where $\mathcal{L}\{f(t)\} = F(s)$

$a \in \mathbb{R}^+$

$f(t) = t\,u(t)$

Integration by parts

$$\int_{0^-}^{\infty} \frac{dv}{dt}\, u\, dt = (u \cdot v)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v \frac{du}{dt}\, dt$$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} \underbrace{e^{-st}}_{\frac{dv}{dt}} \underbrace{t}_{u}\, dt = \cancelto{0}{\left(\underbrace{t}_{u} \cdot \underbrace{\frac{-1}{s} e^{-st}}_{v}\right)\Big|_{0^-}^{\infty}} - \int_{0^-}^{\infty} \underbrace{\frac{-1}{s} e^{-st}}_{v} \cdot \underbrace{1}_{\frac{du}{dt}}\, dt$$

$$= -\left[\frac{1}{s^2} e^{-st}\right]\Big|_{0^-}^{\infty} = -\left[0 - \frac{1}{s^2}\right]$$

$$\boxed{\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}, \quad \text{if } \operatorname{Re}\{s\} > 0}$$