TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\}$	$\mathbf{F}(\mathbf{s}) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\}$	$\mathbf{F}(\mathbf{s}) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})\,u(t)$	$\frac{1}{(\mathbf{s} + \alpha)(\mathbf{s} + \beta)}$
$u(t)$	$\frac{1}{\mathbf{s}}$	$\sin \omega t\, u(t)$	$\frac{\omega}{\mathbf{s}^2 + \omega^2}$
$tu(t)$	$\frac{1}{\mathbf{s}^2}$	$\cos \omega t\, u(t)$	$\frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t),\, n = 1, 2, \ldots$	$\frac{1}{\mathbf{s}^n}$	$\sin(\omega t + \theta)\, u(t)$	$\frac{\mathbf{s}\sin\theta + \omega\cos\theta}{\mathbf{s}^2 + \omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{\mathbf{s} + \alpha}$	$\cos(\omega t + \theta)\, u(t)$	$\frac{\mathbf{s}\cos\theta - \omega\sin\theta}{\mathbf{s}^2 + \omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(\mathbf{s} + \alpha)^2}$	$e^{-\alpha t}\sin \omega t\, u(t)$	$\frac{\omega}{(\mathbf{s} + \alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t),\, n = 1,2,\ldots$	$\frac{1}{(\mathbf{s} + \alpha)^n}$	$e^{-\alpha t}\cos \omega t\, u(t)$	$\frac{\mathbf{s} + \alpha}{(\mathbf{s} + \alpha)^2 + \omega^2}$

Properties of the Laplace Transform

1. Uniqueness if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

2. Linearity $\mathcal{L}\{f(t) + g(t)\} = F(s) + G(s) = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

$\mathcal{L}\{\alpha f(t)\} = \alpha \mathcal{L}\{f(t)\} = \alpha F(s)$

α is some constant

3. Differentiation $\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} \underbrace{e^{-st}}_{u} \underbrace{\frac{df}{dt}}_{\frac{dv}{dt}} dt = \left(e^{-st} f(t)\right)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} (-s) e^{-st} f(t)\, dt$

$= \left[\emptyset - f(0^-)\right] + s\int_{0^-}^{\infty} e^{-st} f(t)\, dt$

$$\boxed{\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)}$$

Derivative property can be applied recursively

$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s\left[sF(s) - f(0^-)\right] - f'(0^-)$

4. Integration

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \int_{0^-}^{\infty} \underbrace{e^{-st}}_{\frac{dv}{dt}} \underbrace{\left(\int_{0^-}^{t} f(\tau)\,d\tau\right)}_{u} dt$$

$$= \left[\left(\int_{0^-}^{t} f(\tau)\,d\tau\right)\left(\frac{-1}{s}e^{-st}\right)\right]\Bigg|_{0^-}^{\infty} - \int_{0^-}^{\infty}\left(\frac{-1}{s}\right)e^{-st} f(t)\,dt$$

$$= \frac{1}{s}\int_{0^-}^{\infty} e^{-st} f(t)$$

$$\boxed{\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)}$$

Initial and Final Value Theorems

Initial Value Theorem

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt = sF(s) - f(0^-)$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} e^{-st}\frac{df}{dt}\,dt + \cancel{\int_{0^+}^{\infty} e^{-st}\frac{df}{dt}\,dt}^{\to 0}\right] = \lim_{s\to\infty}\left[sF(s) - f(0^-)\right]$$

$$\int_{0^-}^{0^+}(1)\frac{df}{dt}\,dt = \lim_{s\to\infty}\left[sF(s)\right] - f(0^-)$$

$$f(0^+) - \cancel{f(0^-)} = \lim_{s\to\infty}\left[sF(s)\right] \cancel{- f(0^-)}$$

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\left[sF(s)\right]}$$

fastest stuff

highest frequencies

Final Value Theorem

$$\lim_{s\to 0}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt\right] = \lim_{s\to 0}\left[sF(s) - f(0^-)\right]$$

$$f(t\to\infty) - \cancel{f(0^-)} = \lim_{s\to 0}\left[sF(s)\right] - \cancel{f(0^-)}$$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sF(s)\right]}$$

slowest response

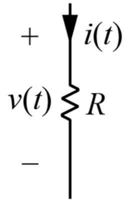
lowest frequencies

TABLE 14.2 Laplace Transform Operations

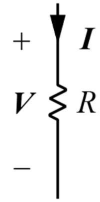
Operation	$f(t)$	F(s)
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(\mathrm{s}) \pm \mathbf{F}_2(\mathrm{s})$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(\mathrm{s})$
Time differentiation	$\dfrac{df}{dt}$	$\mathrm{s}\mathbf{F}(\mathrm{s}) - f(0^-)$
	$\dfrac{d^2f}{dt^2}$	$\mathrm{s}^2\mathbf{F}(\mathrm{s}) - \mathrm{s}f(0^-) - f'(0^-)$
	$\dfrac{d^3f}{dt^3}$	$\mathrm{s}^3\mathbf{F}(\mathrm{s}) - \mathrm{s}^2f(0^-) - \mathrm{s}f'(0^-) - f''(0^-)$
Time integration	$\displaystyle\int_{0^-}^{t} f(t)\,dt$	$\dfrac{1}{\mathrm{s}}\mathbf{F}(\mathrm{s})$
	$\displaystyle\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{1}{\mathrm{s}}F(\mathrm{s}) + \dfrac{1}{\mathrm{s}}\displaystyle\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(\mathrm{s})\mathbf{F}_2(\mathrm{s})$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-a\mathrm{s}}\mathbf{F}(\mathrm{s})$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(\mathrm{s}+\mathrm{a})$
Frequency differentiation	$t f(t)$	$-\dfrac{d\mathbf{F}(\mathrm{s})}{d\mathrm{s}}$
Frequency integration	$\dfrac{f(t)}{t}$	$\displaystyle\int_{s}^{\infty} \mathbf{F}(\mathrm{s})\,d\mathrm{s}$
Scaling	$f(at),\ a \geq 0$	$\dfrac{1}{a}\mathbf{F}\left(\dfrac{\mathrm{s}}{a}\right)$
Initial value	$f(0^+)$	$\lim\limits_{\mathrm{s}\to\infty} \mathrm{s}\mathbf{F}(\mathrm{s})$
Final value	$f(\infty)$	$\lim\limits_{\mathrm{s}\to 0} \mathrm{s}\mathbf{F}(\mathrm{s})$, all poles of $\mathrm{s}\mathbf{F}(\mathrm{s})$ in LHP
Time periodicity	$f(t) = f(t+nT)$, $n = 1, 2, \ldots$	$\dfrac{1}{1-e^{-T_s}}\mathbf{F}_1(\mathrm{s})$, where $\mathbf{F}_1(\mathrm{s}) = \displaystyle\int_{0^-}^{T} f(t)\,e^{-\mathrm{s}t}\,dt$

Circuit Laplace Transform

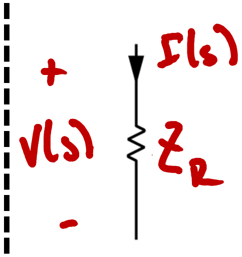
Time Domain	Phasor Domain	s-Domain



$v(t) = i(t)R$



$\mathbf{V} = \mathbf{I}R$

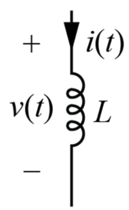


$\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)R\}$

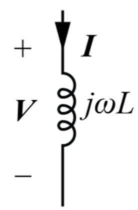
$\boxed{V(s) = I(s) \cdot R}$

$Z_R = R$

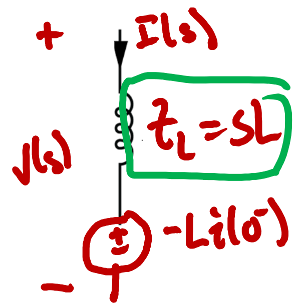
still called "impedance"



$v(t) = L\dfrac{di}{dt}$

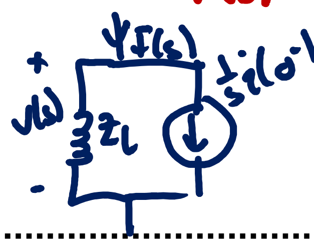


$\mathbf{V} = j\omega L\mathbf{I}$



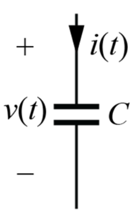
$\mathcal{L}\{v(t)\} = \mathcal{L}\{L\frac{di}{dt}\}$

$V(s) = sL\,I(s) - L\,i(0^-)$

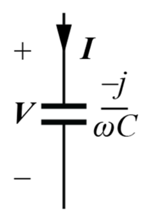


or

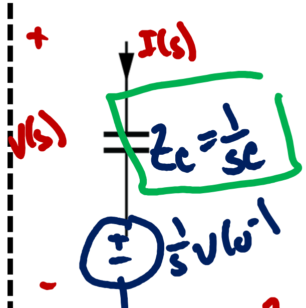
$I(s) = \dfrac{V(s)}{sL} + \dfrac{1}{s}i(0^-)$



$i(t) = C\dfrac{dv}{dt}$

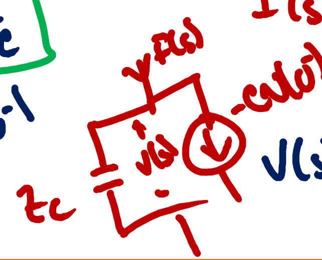


$\mathbf{V} = \dfrac{-j}{\omega C}\mathbf{I}$



$\mathcal{L}\{i(t)\} = \mathcal{L}\{C\frac{dv}{dt}\}$

$I(s) = sC\,V(s) - Cv(0^-)$



or

$V(s) = \dfrac{1}{sC}I(s) + \dfrac{1}{s}v(0^-)$