Complex Roots: Complex Math

ex/ $F(s) = \dfrac{1}{s^2-2s+2} = \dfrac{1}{(s-(1+j))(s-(1-j))}$ roots $e\ \dfrac{2 \pm \sqrt{4-4\cdot 1\cdot 2}}{2} = 1 \pm j$

Complex poles (and their residues) will always show up in complex conjugate pairs for real signals & systems

$F(s) = \dfrac{1}{s^2-2s+2} = \dfrac{k_1}{s-(1+j)} + \dfrac{k_2}{s-(1-j)}$ $\qquad k_2 = k_1^*$

Option 1: Do nothing different

By cover-up method.

$k_1 = \left.\dfrac{1}{s-(1-j)}\right|_{s=1+j} = \dfrac{1}{2j} = \dfrac{-j}{2}$

$k_2 = \left.\dfrac{1}{s-(1+j)}\right|_{s=1-j} = \dfrac{1}{-2j} = \dfrac{j}{2} = k_1^*$ ✓

$f(t) = \mathcal{L}^{-1}\left\{ \dfrac{1/2j}{s-(1+j)} + \dfrac{-\frac{1}{2j}}{s-(1-j)} \right\} = \dfrac{1}{2j} e^{(1+j)t} u(t) + \dfrac{-1}{2j} e^{(1-j)t} u(t)$

$$= \frac{1}{2j}\left[e^{(1+j)t} - e^{(1-j)t}\right]u(t)$$

$$= e^{t}\,\underbrace{\frac{1}{2j}\left[e^{jt} - e^{-jt}\right]}\,u(t)$$

$$\boxed{= e^{t}\sin t\, u(t) = f(t)}$$

Euler:

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Complex Roots: General Case

$$\mathcal{L}^{-1}\left\{ \frac{k}{s-(\sigma+j\omega)} + \frac{k^*}{s-(\sigma-j\omega)} \right\}$$

$$= \left[k e^{(\sigma+j\omega)t} + k^* e^{(\sigma-j\omega)t} \right] u(t)$$

$$k = Re\{k\} + j Im\{k\}$$

$$k^* = Re\{k\} - j Im\{k\}$$

$$= e^{\sigma t}\left[k e^{j\omega t} + k^* e^{-j\omega t} \right] u(t)$$

$$= e^{\sigma t}\left[Re\{k\}\left(e^{j\omega t} + e^{-j\omega t}\right) + j Im\{k\} \underbrace{\left(e^{j\omega t} - e^{-j\omega t}\right)}_{2j\sin\theta} \right] u(t)$$

$$= e^{\sigma t}\left[2 Re\{k\} \cos(\omega t) - 2 Im\{k\} \sin(\omega t) \right] u(t)$$

$$= e^{\sigma t}\left[2\sqrt{Re\{k\}^2 + Im\{k\}^2} \cos\left(\omega t + \tan^{-1}\left(\frac{-Im\{k\}}{Re\{k\}}\right)\right)\right] u(t)$$

$$\boxed{= e^{\sigma t}\, 2|k| \cos(\omega t - \angle k)\, u(t)}$$

Complex Roots: Table Lookup

$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}$$

By tables w/ $\alpha = -1$, $\omega = 1$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = \underline{e^{t}\sin(t)u(t)}$$

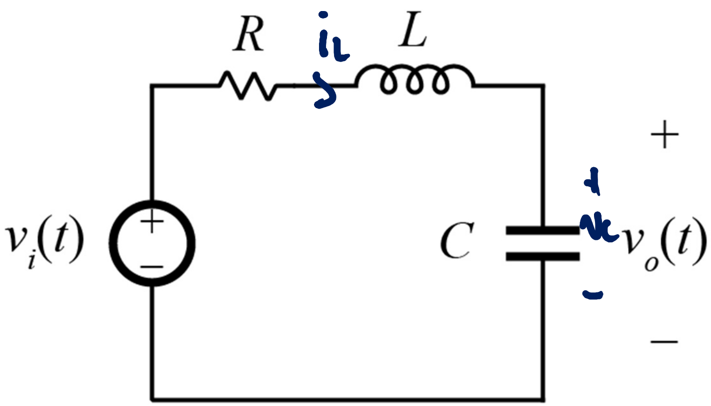
Example

Find $v_o(t)$

$$v_i(t) = \sin(2t)\,u(t)$$

$$L = 500\text{mH},\quad C = 500\text{mF},\quad \text{R} = 2\Omega$$

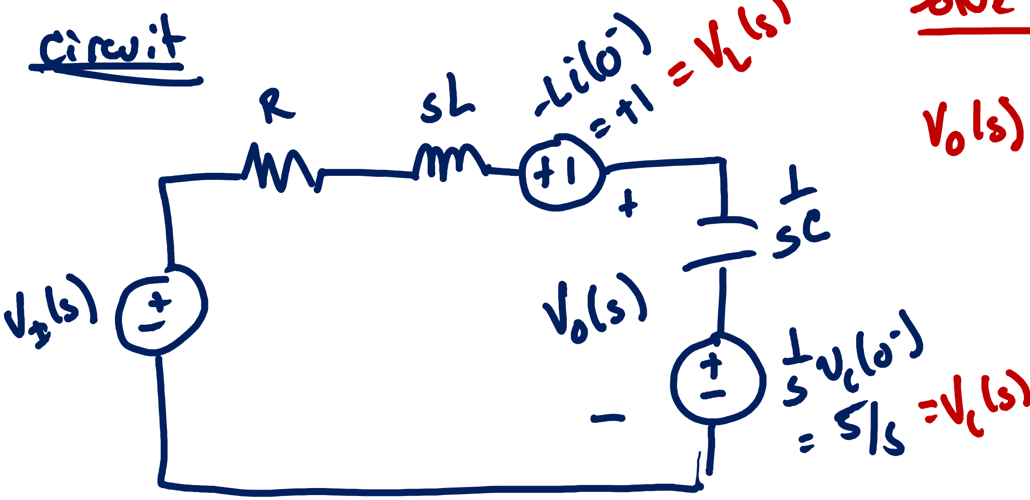
$$v_c(0) = 5\text{V},\quad i_L(0) = -2\text{A}$$



Input

$$V_I(s) = \mathcal{L}\{v_i(t)\} = \frac{2}{s^2+4}$$

Circuit



Solve in s-domain

$$V_o(s) = H_I(s)V_I(s) + H_L(s)V_L(s) + H_C(s)V_C(s)$$

$$H_I(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}$$

$$H_L(s) = (-1)\,H_I(s)$$

$$H_C(s) = (1 - H_I(s))$$

$$\frac{1}{s^2LC + sCR + 1} = \frac{1}{\frac{s^2}{4} + s + 1}$$

$$V_o(s) = \underbrace{\left(\frac{2}{s^2+4}\right)\frac{1}{s^2/4 + s + 1}}_{\text{PFE}} + \underbrace{(-1)\frac{1}{s^2/4 + s + 1}}_{\text{PFE or manipulate}} + \underbrace{\frac{5}{s}}_{\text{tables}} - \underbrace{\frac{5}{s}\,\frac{1}{s^2/4 + s + 1}}_{\text{PFE}}$$