Pole Locations

\[ V_o(s) = V_i(s) H(s) = \left( \sum_{i=0}^{N_1} d_i s^i \right) \left( \sum_{i=0}^{N_H} b_i s^i \right) = \frac{(s - z_1)(s - z_2) \cdots (s - z_{M_H+M_1})}{(s - p_1)(s - p_2) \cdots (s - p_{N_H+N_1})} = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_x}{s - p_x} \]

\[ x = N_H + N_I \]

\[ V_o(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_x e^{p_x t} \]

Poles define the terms in time domain:
- Real poles \( \rightarrow \) exponentials
- Repeated poles \( \rightarrow \) \( t \cdot e^{p_1 t} \)
- Complex poles (pairs) \( \rightarrow \) \( e^{p_1 t} \cos(\omega t + \phi) \)

Some terms in output come from \( H(s) \) independent of \( V_i(s) \):
\[ V_o(s) \uparrow \rightarrow \uparrow H(s) \]

Unilateral Laplace always has some transient at \( t = 0 \)
Unbounded Signals & Unstable Systems

\[ f(t) = e^t \sin(t) e^{-tk} \quad k > 1 \]

Bounded signal \( \exists B \) s.t. |\( f(t) | \leq B \) \( \forall t \)

BIBO stable system: "BIBO" = bounded input, bounded output

Always want any hardware circuit to be bounded & stable
Aside: Laplace and Fourier Revisited

Fourier Transform:

\[ F(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) \, dt \]

Laplace Transform (Bilateral):

\[ F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t) \, dt \]

If we let \( s = j\omega \) Laplace becomes equivalent to Fourier Transform. But some signals don't have a Fourier Transform.

Inverse Fourier Transform:

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t} \, d\omega \]

Inverse Laplace Transform (Bilateral):

\[ f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} \, ds \]
Laplace Explanation

\[ F(s) = \int_{0^-}^{+\infty} e^{-st} f(t) \, dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) \, dt \]

\[ f(t) = \frac{1}{2\pi j} \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} e^{\sigma t} e^{j\omega t} F(s) \, ds \]

**Fourier Series**

Assume we have some function \( f(t) \) which is periodic with period \( T_0 = \frac{2\pi}{\omega_0} \).

\[ f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \]

for \( a_0 \) (constant term)

\[ a_0 = \frac{1}{T_0} \int_{0}^{T_0} f(t) \, dt \]

for \( a_n \): the average value of \( f(t) \cos(n\omega_0 t) \)

\[ a_n = \frac{1}{T_0} \int_{0}^{T_0} f(t) \cos(n\omega_0 t) \, dt \]

for \( b_n \): the average value of \( f(t) \sin(n\omega_0 t) \)

\[ b_n = \frac{1}{T_0} \int_{0}^{T_0} f(t) \sin(n\omega_0 t) \, dt \]

Complex Form of Fourier Series

\[ f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n e^{jnt} + b_n e^{-jnt} \right) \]

\[ f(t) = a_0 + \sum_{n=1}^{\infty} C_n e^{jnt} \]

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Example Signal Laplace Transforms

\[ f(t) = u(t) \]

\[ L[f(t)] = \frac{1}{s} \]

\[ f(t) = e^{-at} \]

\[ L[f(t)] = \frac{1}{s+a} \]

\[ f(t) = e^{-at}u(t) \]

\[ L[f(t)] = \frac{1}{s+a} \]

\[ f(t) = \sin(at) \]

\[ L[f(t)] = \frac{a}{s^2+a^2} \]

\[ f(t) = \cos(at) \]

\[ L[f(t)] = \frac{s}{s^2+a^2} \]

Unstable Signals

- Bounded input, unbounded output (BUU)
- Bounded output, unbounded input (BOU)
- Both bounded, stable (BBS)
- Unbounded, unstable (UBU)

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The s-plane

If all poles are in the open left half-plane, signal/system is Bounded Region of Convergence

\[ F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))} \]

\[ f(t) = e^{t}\sin(t)u(t) \rightarrow \text{in Laplace transform must multiply by } e^{-ot} \text{ for convergence} \]