Eigenvalues of Graphs

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Questions

1. What is the maximum degree centrality of a star graph with $k$ vertices?

2. For a graph with $k$ vertices, how many eigenvalues does the unnormalized graph Laplacian have?

3. Define graph spectrum.
Content

• Graph Centrality
• Graph Laplacians
• Graph Spectrum
• 2-Edge-Covering
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Graph Centrality

Centrality indicates the importance of vertices within a graph.

Its applications include identifying

- Influential scientific papers
- Key infrastructure nodes in a transportation network
- Super-spreaders of disease
- Valuable websites
- etc.
Degree Centrality

Degree centrality is defined as the degree of a vertex.

\[ C_v^{deg} = d_G(v) \]

For directed graph, both indegree and outdegree can be used, depending on the application.
Eigenvector Centrality

Eigenvector centrality is a measure of the influence of a vertex. It’s based on the concept that a vertex is more likely to be influential if it’s connected with influential vertices.

\[ C_v = \frac{1}{\lambda} \sum_{t \in V} A_{v,t} C_t \]

\[ A_{v,t} = \begin{cases} 0 & \text{if } vt \notin E \\ 1 & \text{if } vt \in E \end{cases} \]
Eigenvector Centrality

Rewrite the definition in vector notation

\[ \lambda C = AC \]

\( \lambda \) is an eigenvalue of the adjacency matrix \( A \) and \( C \) is the corresponding eigenvector.
PageRank

PageRank, named after Larry Page, is an algorithm used by Google to rank web pages. PageRank is designed for web pages, but can also be used for general graphs.

Eigenvector centrality is NOT suitable for web pages.
PageRank

Random Jump Assumption:
The user may jump to an arbitrary web page at the probability of $\alpha$

$$Pr_v = \frac{1}{\lambda} \sum_{t \in V} [(1 - \alpha)A_{v,t}Pr_t + \alpha Pr_t]$$

$$\lambda Pr = [(1 - \alpha)A + \alpha E]Pr$$
Unnormalized Graph Laplacian

The unnormalized graph Laplacian is defined as follows

\[ L = D - A \]

\( D \) is the degree matrix and \( A \) is the adjacency matrix.

**Properties:**

- The smallest eigenvalue of \( L \) is 0, and the eigenvector is 1.
- \( L \) has \( n \) non-negative, real-valued eigenvalues
- The multiplicity of the eigenvalue 0 of \( L \), equals to the number of connected components in the graph
Normalized Graph Laplacians

Symmetric normalized Laplacian

$$L^{\text{sym}} = D^{-1/2}LD^{-1/2} = I - D^{-1/2}AD^{-1/2}$$

Random walk normalized Laplacian

$$L^{\text{rw}} = D^{-1}L = I - D^{-1}A$$
Normalized Graph Laplacians

Properties:

• $\lambda$ is an eigenvalue of $L^{rw}$ with eigenvector $u$ iff $\lambda$ is an eigenvalue of $L^{sym}$ with eigenvector $D^{1/2}u$

• $\lambda$ is an eigenvalue of $L^{rw}$ with eigenvector $u$ iff $\lambda$ and $u$ solve $Lu = \lambda Du$

• 0 is an eigenvalue of $L^{rw}$ with 1 as eigenvector. 0 is an eigenvalue of $L^{sym}$ with eigenvector $D^{1/2}1$

• The multiplicity of the eigenvalue 0 of both $L^{rw}$ and $L^{sym}$ equals the number of connected components in the graph
Graph Spectrum

The set of eigenvalues of the adjacency matrix or a Laplacian matrix is called the spectrum of the graph. If two graphs have many eigenvalues in common, that they may also share some structure.

2-Edge-Covering

A graph $G$ is a 2-edge-covering of a graph $H$ if there is an onto map $f : V(G) \rightarrow V(H)$ such that

- if $u-v$ in $G$, then $f(u)-f(v)$ in $H$ and $w(u,v)=w(f(v),f(v))$
- if $f(u)-w$ in $H$, then there is some vertex $v$ in $G$ so that $f(v)=w$
- For each edge $u-v$ in $H$, there are exactly two edges $p-q$ and $r-s$ in $G$ so that $f(p)=f(r)=u$ and $f(q)=f(s)=v$
2-Edge-Covering

$G = \begin{array}{c}
\begin{array}{ccccccc}
3 & 2 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\end{array}$

$H = \begin{array}{c}
\begin{array}{ccccccc}
1/5 & 0 & 2/6 & 3/7 \end{array}
\end{array}$

2-Edge-Covering

The modified cover graph $H^o$ has the same vertices as $H$ but with modification to the weights.

$$w_o(u_0, v_0) = \begin{cases} \sqrt{2}w(u, v) & \text{if $u$ or $v$ is a folding vertex} \\ w(u, v) & \text{otherwise} \end{cases}$$
2-Edge-Covering

For each $v \in G$, define $sgn(v)$ as follows

$$sgn(v) \in \{-1, 0, 1\}$$

$$sgn(v) = \begin{cases} 
0 & \text{if } v \text{ is a folding vertex} \\
-sgn(u) & \text{otherwise, for } u \neq v \text{ and } f(u) = f(v)
\end{cases}$$
2-Edge-Covering

The anti-cover graph $H^o$ is constructed by removing folding vertices, for remaining edge $u^o-v^o$ covered by $u-v$, the weight is

$$w^o(u^o, v^o) = w(u, v)sgn(u)sgn(v)$$

$$H^o = \begin{array}{c}
\bullet & \bullet & \bullet \\
-1 & -1 & -1 \\
-1 & \circ & -1 \\
\end{array}$$
2-Edge-Covering

Theorem:
If $G$ has a 2-edge-covering of $H$, then the spectrum of $G$ is the union of the spectrum of the modified cover graph and the anti-cover graph, counting multiplicity.

The spectrum refers to the eigenvalues of adjacency matrix or normalized Laplacian matrix.
References

Thank you!