Intro to Knot Theory

Redefining Knots as Graphs

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Questions

1. Describe a fundamental problem of Knot Theory.
2. How is Knot Theory used today?
3. Describe a method of transforming a Knot Diagram into two Dual Graphs.
Outline

Introduction

What Is Knot Theory?

A Brief History

Tutte Polynomial

More Definitions
Introduction
Liz
(Dulac)
Bonus: Food
Bonus: Food
Bonus: Food
Bonus: Food
End Intro
What Is Knot Theory?
Definitions

Knot
A knot is a simple closed polygonal curve in $\mathbb{R}^3$

Link
A link is a disjoint union of a finite number of simple closed curves in $\mathbb{R}^3$

Note: A knot is a link with only one component.
Note: Knot Diagrams are not Unique

(i) → I, II → 2 III's → II, I
Legal Transformations

Reidemeister Moves
Kurt Reidemeister (1927) and, independently, James Waddell Alexander and Garland Baird Briggs (1926)
Can we simplify further?

- Solve by hand
- Use invariants to characterize knot
- ...but no known general procedure
Who Cares About Knot Theory?
A Brief History
A Brief History

Early Knot Theory Overview

- Sailing, Climbing, (Theatre), Spiritual, Aesthetic
- Gauss Linking Integral (1833), Listing’s Knot
- Sir William Thomson 1867
A Brief History

Set the Scene

- Introductions to atomic theory 1800s
- Hot topic: What does an atom look like?
- Note: subatomic particles discovered late 19th century
Smoke Ring Atoms

- Peter Tait accidentally inspires vortex atom theory
- Attempt to tabulate all knots leads to Tait Conjectures
- ...turns out atoms aren’t knots
Again... Who Cares About Knot Theory?
Modern Users of Knot Theory

- Math Theorists
- Chemists
- Micro-Biologists
- Quantum Physicists
Where’s the Graph Theory??
Tutte Polynomial
Knot Diagrams to Graphs
Knot Diagrams to Graphs

Knot Invariants

Tutte Polynomial is invariant for knots
Definition: 2-variable Tutte polynomial

\[ T_G = \begin{cases} 
  x(T_{G-e}) & \text{if } e \text{ is a bridge} \\
  y(T_{G-e}) & \text{if } e \text{ is a loop} \\
  T_{G-e} + T_{G/e} & \text{otherwise}
\end{cases} \]

Notation:

- Edge deletion: \( G - e = (V, E - e) \)
- Edge contraction: \( G/e \)
Tutte Polynomial Example

\[ x^2 + x + xy + y + y^2 \]
Some Properties

• If graph $M$ consists of two graphs $M_1$ and $M_2$ with exactly one vertex in common, then $T_M = (T_{M_1})(T_{M_2})$

• If graph $G$ is a dual of $H$, then $T_G(x, y) = T_H(y, x)$
More Definitions
Given a knot (or link) $K$, the Kauffman polynomial is given as...

$$F(K)(a, z) = a^{-w(K)}L(K)$$

where $s(K)$ is the writhe of $K$, and $L(K)$ is the regular isotopy invariant defined by the skein relations.
Intrinsically Knotted and Linked Graphs

**Definition:**

A graph $G$ is intrinsically knotted if in every embedding of $G$, one of the cycles is a non-trivial knot. A graph $G$ is intrinsically linked if in every embedding of $G$, two of the cycles form a non-split link.
Intrinsically Knotted and Linked Graphs

$K_6$: Intrinsically Linked

$K_7$: Intrinsically Knotted
Observe:

If $H$ is a minor of $G$ and $H$ is intrinsically knotted (or linked), then $G$ is intrinsically knotted (or linked) as well.
Theorem:
The Petersen family is the set of minimal minors for the intrinsically linked graphs.
Theorem:
Let $G$ be a graph. Then $(G + 1) + 1$ is intrinsically knotted if and only if $G$ is non-planar.
Questions

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1. Przytycki, Jozef H. KNOTS From Combinatorics of Knot Diagrams to Combinatorial Topology Based on Knots.
2. Barsotti, Jamison. Mattman, Thomas W. Intrinsically Knotted Graphs with 21 Edges
3. Bernstein, Gilbert. When is a Graph Knotted
4. Foisy, Joel S. Ludwig, Lewis D. When graph theory meets knot theory
5. Moreira, Joel. What is... an Intrinsically Knotted Graph?
6. Petersen, David Alan. Tutte Polynomial in Knot Theory
Links to Applications of Knot Theory

1. The Braid Index of Complicated DNA Polyhedral Links
2. Hydrocarbon links in an octet truss
3. KnotProt 2.0: A database of proteins with knots, slipknots, and knotoids
Bonus: Some Interesting Videos

1. What is a Knot? - Numberphile
2. Prime Knots - Numberphile
3. Untangling the mechanics of knots
5. Quantum Money from Knots
6. Knots and Quantum Theory - Edward Witten