Communicating Correlated Sources Over a MAC in the absence of a Gács-Körner Common Part

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Hosted at Texas A & M University.

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Problem Setup: Transmitting Correlated sources over a MAC

Shannon-theoretic study

?? Optimal coding technique ??
?? Necessary and sufficient conditions (in terms of $W_{S_1S_2}, W_{Y|X_1X_2}$) ??

Plain Vanilla Lossless source-channel coding over 2-user MAC.
Problem Setup: Transmitting Correlated sources over a MAC

Shannon-theoretic study

- Optimal coding technique
- Necessary and sufficient conditions (in terms of $W_{S_1S_2}, W_{Y|X_1X_2}$)

Plain Vanilla Lossless source-channel coding over 2-user MAC.

1. A new coding technique.
2. Characterize performance, derive new sufficient conditions.
Prior work: Cover, El Gamal and Salehi (CES) [Nov. 1980] technique. Dueck’s example

\[ H(S_1|S_2) < I(X_1; Y|X_2, S_2, U), \quad H(S_2|S_1) < I(X_2; Y|X_1, S_1, U) \]
\[ H(S_1, S_2|K) < I(X_1X_2; Y|U), \quad H(S_1, S_2) < I(X_1X_2; Y) \]

for a valid pmf \( W_{S_1S_2pU}p_{X_1|US_1}p_{X_2|US_2}W_{Y|X_1X_2} \).

Dueck [Mar 1981] proves single-letter CES technique is sub-optimal (in general) via an ingenious coding technique designed for a particular example.
Main Contribution

1. Build on Dueck’s finding and propose a concatenated coding technique for a general problem instance.

2. Analyse its performance and derive a set of sufficient conditions.

3. Derived set of sufficient conditions can be strictly less binding than those of CES.
Concatenated coding technique with a fixed block-length inner code

Conventional coding technique

A single code mapping the entire chunk of source symbols to a channel codeword. Block-length $n$ tends to infinity.
Concatenated coding technique with a fixed block-length inner code

Proposed concatenated coding technique

Source blocks mapped to codewords via a fixed block-length code irrespective of desired probability of error

\[ S_{1}(1,1) \quad S_{1}(1,2) \quad S_{1}(1,l) \quad S_{1}(2,1) \quad S_{1}(2,2) \quad S_{1}(2,l) \quad S_{1}(m,1) \quad S_{1}(m,2) \quad S_{1}(m,l) \]

\[ l = \text{Block-length of inner code remains fixed} \]

\[ U_{1}(1,1) \quad U_{1}(1,2) \quad U_{1}(1,l) \quad U_{1}(2,1) \quad U_{1}(2,2) \quad U_{1}(2,l) \quad U_{1}(2,1) \quad U_{1}(2,2) \quad U_{1}(2,l) \]

Outer-code of clock-length ml operating on inner code blocks.

\[ X_{1}(1,1) \quad X_{1}(1,2) \quad X_{1}(1,l) \quad X_{1}(2,1) \quad X_{1}(2,2) \quad X_{1}(2,l) \quad X_{1}(m,1) \]
Outline

1. Motivate the concatenated coding structure with a fixed (finite) block-length inner code.
   - Dueck’s [1981] ingenious example and his proposed coding technique.

2. Stitching together single-letter coding techniques appropriately.
   - Effect an inner-outer code structure.
   - Amenable for performance characterization.

3. Characterizing performance of proposed coding technique.

4. Future Directions.
Begin with a very simple example

\[ S_1, S_2 \equiv \text{Sources. Source alphabet} = \{0, 1, \cdots, J\}. \]

Channel inputs \( \equiv X_1, X_2 \in \{0, 1, \cdots, L - 1\} \)

Channel output \( \equiv Y \in \{0, 1, \cdots, L - 1, *\} \).
Begin with a very simple example

Source alphabet = \{0, 1, \cdots, J\}. Source PMF

\[
W_{S_1,S_2} \begin{bmatrix} 0 & 1 & J-1 & J \\ 0 & 1-\varepsilon & 0 & \cdots & 0 & 0 \\ 1 & 0 & \frac{\varepsilon}{J} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\ J-1 & 0 & 0 & \frac{\varepsilon}{J} & 0 \\ J & 0 & 0 & 0 & \frac{\varepsilon}{J} \end{bmatrix}
\]

\[
\begin{array}{c}
0 \quad S_1 \quad 1-\varepsilon \quad S_2 \\
0 \quad 1 \quad \frac{\varepsilon}{J} \\
J-1 \quad J-1 \quad \frac{\varepsilon}{J} \\
J \quad J \quad \frac{\varepsilon}{J} \\
\end{array}
\]
Begin with a very simple example

Source alphabet = \{0, 1, \cdots, J\}. Source PMF

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W_{S_1,S_2} \begin{bmatrix}
0 & 1 & | & J-1 & J \\
0 & 1-\varepsilon & 0 & \cdots & 0 & 0 \\
1 & 0 & \frac{\varepsilon}{J} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
J-1 & 0 & 0 & \frac{\varepsilon}{J} & 0 \\
J & 0 & 0 & 0 & \frac{\varepsilon}{J}
\end{bmatrix}
\]

\[S_1 = S_2 \text{ w.p. } 1.\]
Begin with a very simple example

Source alphabet = \{0, 1, \cdots, J\}. Source PMF

\[
\begin{array}{c|c|c|c|c}
W_{S_1, S_2} & 0 & 1 & J-1 & J \\
\hline
0 & 1-\varepsilon & 0 & \cdots & 0 & 0 \\
1 & 0 & \frac{\varepsilon}{J} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
J-1 & 0 & 0 & \frac{\varepsilon}{J} & 0 \\
J & 0 & 0 & 0 & \frac{\varepsilon}{J} \\
\end{array}
\]

\[S_1 = S_2 \text{ w.p } 1. \text{ } H(S_1) = h_b(\varepsilon) + \varepsilon \log J\]
Begin with a very simple example

Source alphabet $= \{0, 1, \cdots, J\}$. Source PMF

$W_{S_1, S_2}$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>$J-1$</th>
<th>$J$</th>
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<tr>
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</tbody>
</table>

$S_1 = S_2$ w.p. 1. $H(S_1) = h_b(\epsilon) + \epsilon \log J$

MAC Channel

Noiseless if $X_1 = X_2$ and Erasure if $X_1 \neq X_2$
Begin with a very simple example

Source alphabet = \{0, 1, \cdots, J\}. Source PMF

\[
\begin{array}{c|cccc}
W_{S_1,S_2} & 0 & 1 & J-1 & J \\
\hline
0 & 1-\varepsilon & 0 & \cdots & 0 & 0 \\
1 & 0 & \frac{\varepsilon}{J} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
J-1 & 0 & 0 & \frac{\varepsilon}{J} & 0 \\
J & 0 & 0 & 0 & \frac{\varepsilon}{J} \\
\end{array}
\]

\[S_1 = S_2 \text{ w.p. } 1. \quad H(S_1) = h_b(\epsilon) + \epsilon \log J\]

MAC Channel

\[X_1, X_2, Y \in \{0, 1, \ldots, L-1\}\]
\[Y = X_1 = X_2 \text{ if } X_1 = X_2 \]
\[Y = * \text{ if } X_1 \neq X_2\]
Begin with a very simple example

Source alphabet $= \{0, 1, \cdots, J\}$. Source PMF

<table>
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</tbody>
</table>

$S_1 = S_2$ w.p 1. $H(S_1) = h_b(\varepsilon) + \varepsilon \log J$

Squeeze $\varepsilon$, blow up $J$ and yet have $\log(L) = H(S_1)$. 

MAC Channel

$X_1, X_2, Y \in \{0, 1, \ldots, L-1\}$

$Y = X_1 = X_2$ if $X_1 = X_2$

$Y = *$ if $X_1 \neq X_2$
Begin with a very simple example

Source alphabet = \{0, 1, \cdots, J\}. Source PMF

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<td>J</td>
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MAC Channel

X_1, X_2, Y \in \{0, 1, \ldots, L-1\}

Y = X_1 = X_2 if X_1 = X_2

Y = * if X_1 \neq X_2

S_1 = S_2 w.p 1. \(H(S_1) = h_b(\varepsilon) + \varepsilon \log J\)

Squeeze \varepsilon, blow up \(J\) and yet have \(\log(L) = H(S_1)\). Source is transmissible.
Begin with a very simple example

Source alphabet $= \{0, 1, \cdots, J\}$. Source PMF

$$W_{S_1, S_2} = \begin{bmatrix} 0 & 1 & J-1 & J \\ 0 & 1-\epsilon & 0 & \cdots & 0 & 0 \\ 1 & 0 & \frac{\epsilon}{J} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J-1 & 0 & 0 & \frac{\epsilon}{J} & 0 \\ J & 0 & 0 & 0 & \frac{\epsilon}{J} \end{bmatrix}$$

MAC Channel

$S_1 = S_2$ w.p 1. $H(S_1) = h_b(\epsilon) + \epsilon \log J$

Squeeze $\epsilon$, blow up $J$ and yet have $\log(L) = H(S_1)$. Source is transmissible.

Need $X_1 = X_2$ uniform and $X_1 = X_2$.

For small $\epsilon$, source highly non-uniform.
Identical maps of large block-length can ensure $X_1 = X_2$ uniform

With $\log L = H(S_1)$, to transmit sources we need

$X_1 = X_2$ uniform and $X_1 = X_2$.

Since $S_1 = S_2$, identical maps $S_1 \to X_1$ and $S_2 \to X_2$ imply $X_1 = X_2$.
Identical maps of large block-length can ensure $X_1 = X_2$ uniform

With $\log L = H(S_1)$, to transmit sources we need

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Since $S_1 = S_2$, identical maps $S_1 \rightarrow X_1$ and $S_2 \rightarrow X_2$ imply $X_1 = X_2$

Map large (typical) source sequences to codewords chosen uniformly at random.

We can ensure

$X_1 = X_2$ uniform and $X_1 = X_2$. 

How many source symbols determine each channel symbol?

To ensure $X_1$ and $X_2$ are uniform, we need to wait for the source sequence to become uniform on its typical set.

Smaller the $\epsilon$, longer it takes for the source to become uniform on its typical set.

Smaller the $\epsilon$, larger the block-length.

For small $\epsilon$, to ensure $X_1$ and $X_2$ are uniform, each channel symbol is chosen as a (non-trivial) fn. of many source symbols.
What if we tweak the source just a little?

$$W_{S_1,S_2}$$

<table>
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[Diagram showing a graph with nodes and edges labeled with $\varepsilon/J$]
What if we tweak the source just a little?

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<tr>
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![Diagram](image-url)
What if we tweak the source just a little?

\[
\begin{array}{cccccc}
W_{s_1,s_2} & 0 & 1 & \cdots & J-1 & J \\
0 & 1-\varepsilon & \theta & \cdots & \theta & \theta \\
1 & 0 & \frac{\varepsilon}{J-\theta} & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
J-1 & 0 & 0 & \frac{\varepsilon}{J-\theta} & 0 & \\
J & 0 & 0 & 0 & \frac{\varepsilon}{J-\theta} & \\
\end{array}
\]

MAC Channel remains same.

\[X_1 \xrightarrow{\text{square}} Y \xrightarrow{\text{square}} X_2\]

\(X_1, X_2, Y \in \{0, 1, \ldots, L-1\}\)

\(Y = X_1 = X_2 \text{ if } X_1 = X_2\)

\(Y = * \text{ if } X_1 \neq X_2\)
What if we tweak the source just a little?

MAC Channel remains same.

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</table>

Source stripped off its Gács-Körner part.

\[ P(S_1 \neq S_2) = J\theta \implies P(S_1^n \neq S_2^n) = 1 - (1 - J\theta)^n \xrightarrow{n \to \infty} 1. \]

With increasing block-length $n$, $P(X_1^n \neq X_2^n) \to 1$. 

$X_1 \xrightarrow{Y} X_2$

$X_1, X_2, Y \in \{0, 1, \ldots, L-1\}$

$Y = X_1 = X_2$ if $X_1 = X_2$

$Y = *$ if $X_1 \neq X_2$
What if we tweak the source just a little?

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</table>

MAC Channel remains same.

$X_1 \rightarrow \text{Y} \rightarrow X_2$

$X_1, X_2, Y \in \{0, 1, \ldots, L-1\}$

$Y = X_1 = X_2$ if $X_1 = X_2$

$Y = *$ if $X_1 \neq X_2$

Large block-length necessary to make chnl. inputs uniform.

Short block-length necessary to make $X_1 = X_2$ with high probability.
What if we tweak the source just a little?

MAC Channel remains same.

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</table>

Building around this, and carefully choosing values for $\epsilon, \theta, J, L$, Dueck [1981] proves


2. Mapping fixed length blocks of source to channel codewords is more efficient.
Why is single-letter Cover, El Gamal, Salehi (CES) technique sub-optimal?

If $S_1, S_2$ have a GK common part $K$, 

$S_1$

$K$

$S_2$

$X_1$

$X_2$

2-MAC

$W_{Y|X_1X_2}$

$Y$

$Rx$

$S_1S_2$
Why is single-letter Cover, El Gamal, Salehi (CES) technique sub-optimal?

If $S_1, S_2$ have a GK common part $K$, the Cover, El Gamal and Salehi technique can induce a common part $U$ on the chnl.
Why is single-letter Cover, El Gamal, Salehi (CES) technique sub-optimal?

If $S_1$, $S_2$ have a GK common part $K$, the Cover, El Gamal and Salehi technique can induce a common part $U$ on the chnl.

However, in the absence of GK common part, Cover, El Gamal and Salehi technique is constrained to the single-letter (long) Markov chain $X_1 - S_1 - S_2 - X_2$. 
Why is single-letter Cover, El Gamal, Salehi (CES) technique sub-optimal?

If $S_1, S_2$ have a GK common part $K$, the Cover, El Gamal and Salehi technique can induce a common part $U$ on the chnl.

However, in the absence of GK common part, Cover, El Gamal and Salehi technique is constrained to the single-letter (long) Markov chain $X_1 - S_1 - S_2 - X_2$.

By mapping short blocks of the source, Dueck is able to induce a distribution on the channel inputs that is closer to the requirement $X_1 = X_2$ uniform and $X_1 = X_2$ than that permitted by the single-letter (long) Markov chain $X_1 - S_1 - S_2 - X_2$. 
Why is single-letter Cover, El Gamal, Salehi (CES) technique sub-optimal?

Dueck’s technique being multi-letter is NOT constrained by the single-letter (long) Markov chain $X_1 - S_1 - S_2 - X_2$, but is only constrained by a multi-letter long Markov chain $X_1 - S_{1l} - S_{2l} - X_2$
Fixed block-length coding forces a concatenated coding scheme

**Source PMF**

<table>
<thead>
<tr>
<th>(W_{S_1,S_2})</th>
<th>0</th>
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Block-length of coding is determined by \(\epsilon, \theta, J, L\), not be the desired prob. of error.

This fixed block-length code results in non-vanishing probability of error.

Necessitates an outer code of arbitrarily large block-length to

1. Correct for errors in the fixed block-length code.
2. Communicate rest of information.

This results in **concatenated coding structure**.

**MAC Channel**

\(X_1, X_2, Y \in \{0, 1, \ldots, L-1\}\)

\(Y = X_1 = X_2\) if \(X_1 = X_2\)

\(Y = *\) if \(X_1 \neq X_2\)
Part 2 and 3

Concatenated coding scheme for a general problem instance whose info-theoretic performance can be characterized?
What challenges does the finite block-length code throw up?

Revisit CES scheme with common part $K$

Two phase coding, in the presence of GK common part $K$.

Phase 1: Common part $K^n$ is decoded perfectly (owing to infinite block-length)
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Two phase coding, in the presence of GK common part $K$.

Phase 1: Common part $K^n$ is decoded perfectly (owing to infinite block-length)

Phase 2: Rest of Information: $S_1^n, S_2^n$ conditioned on common part $K^n$ is encoded in $X_1^n, X_2^n$ and decoded by Rx.

$K^n$ can be viewed as side-information available at all terminals.
What challenges does the finite block-length code throw up?

Suppose $S_1, S_2$ do NOT have common part, but highly correlated.

$m \times l$ matrix encoding. $l$ remains fixed, $m \to \infty$.

Two phase coding.

Phase 1: Rows of $K_1^{m \times l}$ ($K_2^{m \times l}$) mapped to $U_1^{m \times l}$ ($U_2^{m \times l}$) via finite block-length code. Decoded erroneously as $V^{m \times l}$. 
What challenges does the finite block-length code throw up?

Suppose \( S_1, S_2 \) do NOT have common part, but highly correlated.

\[
\begin{align*}
S_1^{m \times l} & \quad K_1^{m \times l} & \quad U_1^{m \times l} & \quad X_1^{m \times l} & \quad 2-\text{MAC} & \quad Y^{m \times l} & \quad \text{Rx} & \quad V^{m \times l} & \quad G^{m \times l} & \quad S_1^{m \times l}S_2^{m \times l} \\
S_2^{m \times l} & \quad K_2^{m \times l} & \quad U_2^{m \times l} & \quad X_2^{m \times l}
\end{align*}
\]

\( m \times l \) matrix encoding. \( l \) remains fixed, \( m \to \infty \).

Two phase coding.

Phase 1: Rows of \( K_1^{m \times l} (K_2^{m \times l}) \) mapped to \( U_1^{m \times l} (U_2^{m \times l}) \) via finite block-length code. Decoded erroneously as \( V^{m \times l} \).

Phase 2: Rest of Information: \( S_1^{m \times l}, S_2^{m \times l} \) conditioned on common part \( V^{m \times l} \) is encoded in \( X_1^{m \times l}, X_2^{m \times l} \) and decoded by Rx.
Multi-letter distribution induced by the fixed block-length code

Each row of matrices $U_1^{m \times l}$, $U_2^{m \times l}$, $V^{m \times l}$, $G^{m \times l}$ have a multi-letter pmf. How do you perform superposition coding?

Leverage the technique of interleaving suggested in Shirani and Pradhan [ISIT 2014].

Note that the inner code of finite block-length is fixed upfront.

As a consequence, the rows of each of the $m \times l$ matrices are iid. However, each row has a product distribution.
What is Interleaving?
Interleaving [Shirani and Pradhan 2014]

\[
\begin{array}{ccc}
E(1,1) & E(1,2) & E(1,1) \\
E(t,1) & E(t,2) & * & * & * \\
E(m,1) & E(m,2) & E(m,1) \\
\end{array}
\]

\[\sim \prod_{E} p_{E}^{l}\]

\[E(1,1;l) \ldots E(m,1;l) \sim \prod_{E} p_{E}^{l}\]
### Interleaving [Shirani and Pradhan 2014]

\[
\begin{array}{ccc}
E(1,1) & E(1,2) & \cdots & E(1,l) \\
E(t,1) & E(t,2) & \cdots & E(t,l) \\
E(m,1) & E(m,2) & & E(m,l) \\
\end{array}
\]

\[\sim \prod_{E_l} \]

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be unif. And randomly chosen permutations in $\{1, 2, \ldots, l\}$
The same argument holds for each sub-vectors
$E(1, \lambda_1(i)) \cdots E(m, \lambda_m(i)) : i = 1, \ldots, l.$

We therefore have $l$ iid sub-vectors $E(1, \lambda_1(i)) \cdots E(m, \lambda_m(i)) : i = 1, \ldots, l.$
A new coding theorem in a simplified form

Theorem

A pair of sources \((S, W_S)\) is transmissible over a MAC \((X, Y, W_{Y|X})\) if there exists (i) a finite set \(K\), maps \(f_j : S_j \rightarrow K\), with \(K_j = f_j(S_j)\) for \(j = 1, 2\), (ii) finite set \(U\) and pmf \(p_{U|X|U}\) defined on \(U \times X\), (iii) \(\delta > 0, l \in \mathbb{N}, \rho \in [0, 1]\), (iv) \(A, B \geq 0\) such that

\[
A + B \geq (1 + \delta)H(K_1), \text{ and for } j \in [2],
H(S_j|S_j, K) + \theta < I(X_j; Y, X_j|U) - h_b(\phi) - \phi \log |U|
- |X_j||Y||U|(1 + |X_j|)\phi \log \frac{1}{\phi}, \quad (1)
\]

\[
B + H(S|K) + \theta < H(X_1|U) + H(X_2|U) - H(X|Y, U)
- 2h_b(\phi) - 2\phi \log |U| - (1 + |X|)|Y||U|\phi \log \frac{1}{\phi}, \quad \phi \in [0, 0.5), \quad (2)
\]

where \(\phi = g_{\rho, l} + \xi^l + \tau_{l, \delta}\) where \(\xi^l := 1 - (1 - \xi)^l\),

\(\xi = P(K_1 \neq K_2), \tau_{l, \delta} := 2|K| \exp\{-2\delta^2 p_{K_1}(a^*)l\}\) and

\(\theta = h_b(\phi) + 2\phi \log |K|,\)

\(g_{\rho, l} = g_{\rho, l} = \exp\{-l[E_r(A + \rho, p_U, p_{Y|U}) - \rho]\}, \text{ where}\)

\(E_r(R, p_U, W_{Y|U}) = \min_{V_{Y|U}} \{D(V_{Y|U}||W_{Y|U}|p_U) + |I(p_U; V_{Y|U}) - R|^+\}. \quad (3)\)
Backup
Finite block-length inner code concatenated with an large block-length outer code

Let \( U_l \) be \( C_U \) codeword chosen by encoder \( j \).

\[
P(U_1^l \neq U_2^l) \leq P(K_1^l \neq K_2^l) + P(K_1^l \notin T_{\delta}(K_1))
\leq 1 - (1 - \xi)^l + 2|\mathcal{K}| \exp\{-2p_{K_1}^2(a^*)l\} =: \beta
\]

Suppose \( g \) is the prob. of error of a PTP channel decoder operating over a PTP channel \( p_{Y|U} \).
Inner block code of finite block length

Assuming encoders agree, decoder decodes into $C_U$-codebook with a PTP decoder on $p_{Y|U}$ chnl.
Inner block code of finite block length

$V$ is decoded version of $U$. $G(t, 1 : l) = e_U^{-1}(V(t, 1 : l))$ is decoded version of $K$. 
Encoders and decoder agree on a fraction $1 - (g + \beta)$ of the rows of $U$-matrices.
Outer code: The approach

Encoder 1 has

\[ S_1(1 : l) \cdots S_1((m - 1)l + 1 : ml) \]
\[ K_1(1 : l) \cdots K_1((m - 1)l + 1 : ml) \]
\[ U_1(1 : l) \cdots U_1((m - 1)l + 1 : ml) \]

Encoder 2 has

\[ S_2(1 : l) \cdots S_2((m - 1)l + 1 : ml) \]
\[ K_2(1 : l) \cdots K_2((m - 1)l + 1 : ml) \]
\[ U_2(1 : l) \cdots U_2((m - 1)l + 1 : ml) \]

Decoder has

\[ G(1 : l) \cdots G((m - 1)l + 1 : ml) \]
\[ V(1 : l) \cdots V((m - 1)l + 1 : ml) \]

If these had a single-letter form i.e., distributed as \( \prod_{t=1}^{ml} p_{S_1K_1U_1S_2K_2U_2V} \), then one could apply standard info-theoretic techniques.
Challenges

Encoder 1 has
\[ S_1(1 : l) \cdots S_1((m - 1)l + 1 : ml) \]
\[ K_1(1 : l) \cdots K_1((m - 1)l + 1 : ml) \]
\[ U_1(1 : l) \cdots U_1((m - 1)l + 1 : ml) \]

Encoder 2 has
\[ S_2(1 : l) \cdots S_2((m - 1)l + 1 : ml) \]
\[ K_2(1 : l) \cdots K_2((m - 1)l + 1 : ml) \]
\[ U_2(1 : l) \cdots U_2((m - 1)l + 1 : ml) \]

Decoder has
\[ G(1 : l) \cdots G((m - 1)l + 1 : ml) \]
\[ V(1 : l) \cdots V((m - 1)l + 1 : ml) \]

- Not guaranteed to have a iid form.
- Even if we can extract sub-vectors that have product iid form, we do not know the underlying pmf. Hence, cannot express rates in terms of these pmfs.
The rows are independent and identically distributed.

The rows are independent.
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\[
\begin{array}{cccc}
S_1(t,1) & S_1(t,2) & \cdots & S_1(t,l) \\
K_1(t,1) & K_1(t,2) & \cdots & K_1(t,l) \\
U_1(t,1) & U_1(t,2) & \cdots & U_1(t,l) \\
S_2(t,1) & S_2(t,2) & \cdots & S_2(t,l) \\
K_2(t,1) & K_2(t,2) & \cdots & K_2(t,l) \\
U_2(t,1) & U_2(t,2) & \cdots & U_2(t,l) \\
V(t,1) & V(t,2) & \cdots & V(t,l) \\
G(t,1) & G(t,2) & \cdots & G(t,l) \\
\end{array}
\]

\[
E(t,1:l) \sim \rho_E^l
\]

\[
E(1,1:l) \ldots E(m,1:l) \sim \prod_{l} \rho_E^l
\]

The rows are independent and identically distributed.
Randomly chosen co-ordinates are iid!!!

For each row $t$, choose column $\pi_t(1)$, where $\pi_1, \cdots, \pi_m$ are random permutations.
Randomly chosen co-ordinates are iid!!!
Randomly chosen co-ordinates are iid!!!

Let $\Pi_1, \Pi_2, \ldots, \Pi_m$ be unif. And randomly chosen permutations in $\{1, 2, \ldots, l\}$

$$E(1, \Pi_1(1)) \ E(2, \Pi_2(1)) \ldots \ E(m, \Pi_m(1)) \sim \prod_{t=1 \ldots m} \frac{1}{l} \sum_{i=1 \ldots l} p_{E_i}$$

The rows are independent and identically distributed.

We therefore have $l$ iid sub-vectors $E(1, \pi_1(i)) \cdots E(m, \pi_m(i)) : i = 1, \ldots, l.$
Randomly chosen co-ordinates are iid!!

Co-ordinates with same color coded together within a block in the outer code. $l$ such blocks.
Randomly chosen co-ordinates are iid!!!

Co-ordinates with same color coded together within a block in the outer code. $l$ such blocks.
Randomly chosen co-ordinates are iid!!!
Randomly chosen co-ordinates are iid!!

Co-ordinates with same color coded together within a block in the outer code. $l$ such blocks.
The induced single-letter distributions are unknown!!!

\[ S_1(1, \pi_1(1))S_1(2, \pi_2(1)) \cdots S_1(m, \pi_m(1)) \]
\[ K_1(1, \pi_1(1))K_1(2, \pi_2(1)) \cdots K_1(m, \pi_m(1)) \]
\[ U_1(1, \pi_1(1))U_1(2, \pi_2(1)) \cdots U_1(m, \pi_m(1)) \]
\[ S_2(1, \pi_1(1))S_2(2, \pi_2(1)) \cdots S_2(m, \pi_m(1)) \]
\[ K_2(1, \pi_1(1))K_2(2, \pi_2(1)) \cdots K_2(m, \pi_m(1)) \]
\[ U_2(1, \pi_1(1))U_2(2, \pi_2(1)) \cdots U_2(m, \pi_m(1)) \]
\[ V(1, \pi_1(1))V(2, \pi_2(1)) \cdots V(m, \pi_m(1)) \]
\[ G(1, \pi_1(1))G(2, \pi_2(1)) \cdots G(m, \pi_m(1)) \]

\[ \sim \prod_{t=1}^{m} p \hat{S}_1 \hat{K}_1 \hat{U}_1 \hat{S}_2 \hat{K}_2 \hat{U}_2 \hat{V} \hat{G} \]

The sufficient conditions will look like

\[ H(\hat{S}_1|\hat{S}_2, G) < I(\hat{X}_1; Y \hat{X}_2|\hat{V} \hat{S}_2), \quad H(\hat{S}_2|\hat{S}_1, G) < I(\hat{X}_2; Y \hat{X}_1|\hat{V} \hat{S}_1) \cdots \]

But, we do not know this pmf.
The induced single-letter distributions are unknown!!!

\[
S_1(1, \pi_1(1))S_1(2, \pi_2(1)) \cdots S_1(m, \pi_m(1)) \\
K_1(1, \pi_1(1))K_1(2, \pi_2(1)) \cdots K_1(m, \pi_m(1)) \\
U_1(1, \pi_1(1))U_1(2, \pi_2(1)) \cdots U_1(m, \pi_m(1)) \\
S_2(1, \pi_1(1))S_2(2, \pi_2(1)) \cdots S_2(m, \pi_m(1)) \\
K_2(1, \pi_1(1))K_2(2, \pi_2(1)) \cdots K_2(m, \pi_m(1)) \\
U_2(1, \pi_1(1))U_2(2, \pi_2(1)) \cdots U_2(m, \pi_m(1)) \\
V(1, \pi_1(1))V(2, \pi_2(1)) \cdots V(m, \pi_m(1)) \\
G(1, \pi_1(1))G(2, \pi_2(1)) \cdots G(m, \pi_m(1))
\]

\[\sim \prod_{t=1}^{m} p_{\hat{S}_1 \hat{K}_1 \hat{U}_1 \hat{S}_2 \hat{K}_2 \hat{U}_2 \hat{V} \hat{G}}\]

The sufficient conditions will look like

\[H(\hat{S}_1|\hat{S}_2, G) < I(\hat{X}_1; Y \hat{X}_2|\hat{V} \hat{S}_2), \quad H(\hat{S}_2|\hat{S}_1, G) < I(\hat{X}_2; Y \hat{X}_1|\hat{V} \hat{S}_1) \cdots\]

But, we do not know this pmf. This pmf depends on the pmf of the empirical distribution of the inner layer code and the errors in that layer.
Upperbound the difference using relationship between chosen and actual pmf

Let \( p_U p x_1 | u s_1 p x_2 | u s_2 \) be chosen pmf.

\( p_{\hat{U}_j} = p_U \) since symbols of \( C_U \) chosen iid \( p_U \).

\[
P(\hat{U}_1 \neq \hat{U}_2) \leq \beta
\]

\( p_{x_j | \hat{u}_j \hat{s}_j} = p_{x_j | u s_j} \): By choice of coding technique. One can derive an upper bound using these relations.
Upperbound the difference using relationship between chosen and actual pmf

Let $p_U p_{X_1|U S_1} p_{X_2|U S_2}$ be chosen pmf.

$p_{\hat{U}_j} = p_U$ since symbols of $C_U$ chosen iid $p_U$.

$P(\hat{U}_1 \neq \hat{U}_2) \leq \beta$

$p_{X_j|\hat{U}_j \hat{S}_j} = p_{X_j|U S_j}$: By choice of coding technique. One can derive an upper bound using these relations.