Fixed Block-Length Coding in Network Information Theory

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Asymptotically Large Block-Length (B-L) codes Ubiquitous in Info. Th.

Larger the B-L, Better the performance.
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Larger the B-L, Better the performance.

Efficiency gets better
Asymptotically Large Block-Length (B-L) codes Ubiquitous in Info. Th.

Larger the B-L, Better the performance.

Efficiency gets better and better ....

Processing of Larger blocks is good. Law of Large numbers.

Larger blocks ⇒ Greater efficiency.

Source compression, Quantization, Channel Coding, Inference, Estimation ...
Compression, Chnl. Coding, …, BUT for extracting correlation.

Distributed correlated $S_1$ and $S_2$. $X_1 = f_1(S_1^n)$, $X_2 = f_2(S_2^n)$.

Maximize Correlation $\rho(X_1, X_2)$ among $n$–letter fns.
Compression, Chnl. Coding, · · ·, BUT for extracting correlation.

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Maximize Correlation $\rho(X_1, X_2)$ among $n$–letter fns.
Compression, Chnl. Coding, · · ·, BUT for extracting correlation.

\[ \rho_{100}^{*}(X_1, X_2) > \rho_{1000}^{*}(X_1, X_2) \]

\[ \max \rho(X_1, X_2) \]

Correlation

Distributed correlated \( S_1 \) and \( S_2 \). \( X_1 = f_1(S_1^n), X_2 = f_2(S_2^n) \).

Maximize Correlation \( \rho(X_1, X_2) \) among \( n \)-letter fns.
Compression, Chnl. Coding, ⋅⋅⋅, BUT for extracting correlation.

Distributed correlated $S_1$ and $S_2$. $X_1 = f_1(S_1^n)$, $X_2 = f_2(S_2^n)$.

Maximize Correlation $\rho(X_1, X_2)$ among $n$–letter fns.
Compress, Chnl. Coding, ⋯, BUT for extracting correlation.

\[ S_1 \quad 05284649 \]
\[ \rho^{*}_{10}(X_1, X_2) > \]
\[ p_{S_1 S_2} \quad \rho^{*}_{50}(X_1, X_2) > \quad \rho^{*}_{100}(X_1, X_2) > \rho^{*}_{1000}(X_1, X_2) \]
\[ X_1 = 3 \]
\[ X_2 = 3 \]

Distributed correlated \( S_1 \) and \( S_2 \). \( X_1 = f_1(S_1^n) \), \( X_2 = f_2(S_2^n) \).

Maximize Correlation \( \rho(X_1, X_2) \) among \( n \)-letter fns.

Correlation between Processed data reduces with block-length.

Witsenhausen’s 1-bit extractor [1975].

\[ X_1 = f_1(S_1^n), X_2 = f_2(S_2^n). \text{ Corr. Measure } \rho(X_1, X_2) \text{ decreases with increasing } n. \]
Shorter Blocks

Efficient transfer of correlation.

What if my network problem involves correlation extraction AND compression/Chnl coding?

Longer Block

Efficient Chnl Coding, Source Compression.
Shorter Blocks

Efficient transfer of correlation.

Longer Block

Efficient Chnl Coding, Source Compression.

What if my network problem involves correlation extraction AND compression/Chnl coding?

A tension in choice of Optimal B-L

Examples include Some Fundamental Long-Standing NIT problems
Joint Source-Channel Coding over 2–MAC and 2–IC

Studied since 1960s. Slepian, Wolf; Han; Cover, El Gamal, Salehi · · ·

Extracting Correlation enables **Co-ordination** among **distributed Txs**.
Distributed Source Coding (Berger-Tung problem)

Correlation can enable distributed compression.
Robust Distributed Source Coding [Chen, Berger 2009]

Distributed source coding + multiple descriptions coding.
Certain Codes have to have **Fixed B-L**

involve codes that need to

**extract correlation**  AND **compress/beat chnl. noise**

**B-L of these codes : Neither too big, Nor too small. Fixed B-L codes**
Our Contributions

Conventional $\infty-B-L$ coding schemes are strictly sub-optimal.

Designed New coding schemes employing certain codes of fixed B-L.

Info.-theoretic analysis of codes with fixed B-L.

Correlated Sources Over MAC, IC. Robust Distributed Source Coding.

New Coding Structure for

that strictly outperform all previous known.

Derived new sufficient conditions, achievable rate-dist. regions.

MAC and IC problems have remained Stubborn since 1981.
Robustness and Compression

Multiple Description Coding

$w_x \sim X$ → Encoder → Decoder 1

$\hat{X}_1 \sim \Delta_1$

$R_1$

Decoding 1

Emphasizes Robustness.

Incorporate Robustness and Compression in a Distributed Encoder Setup.

Centralized Encoder.
Robustness and Compression

Distributed Source Coding (Berger-Tung Problem)

\[ Y_1 \rightarrow \text{Encoder 1} \rightarrow R_1 \rightarrow \text{Decoder 3} \rightarrow \hat{Y}_1, \hat{Y}_2 \sim \Delta_1, \Delta_2 \]

Emphasizes Compression.

\[ Y_2 \rightarrow \text{Encoder 2} \rightarrow R_2 \rightarrow \text{Decoder 3} \rightarrow \hat{Y}_1, \hat{Y}_2 \sim \Delta_1, \Delta_2 \]
Multiple Description Coding

- Emphasizes Robustness.
- Centralized Encoder.

Distributed Source Coding (Berger-Tung Problem)

- Emphasizes Compression.

Robustness and Compression
Robustness and Compression

Multiple Description Coding

Emphasizes Robustness.
Centralized Encoder.

Distributed Source Coding (Berger-Tung Problem)

Emphasizes Compression.

Incorporate Robustness and Compression in a Distributed Encoder Setup.
Problem Setup: Robust Distributed Source Coding [Chen and Berger]

\[ X \sim \text{Hidden Source.} \]

\[ Y_1, Y_2 \sim \text{Observed Distributed Correlated Sources.} \]

Reconstruct \( X \) at three decoders.

Decoder 1: Fed Bit stream 1. Rate \( R_1 \). Target Distortion \( \Delta_1 \).
Decoder 2: Fed Bit stream 2. Rate \( R_2 \). Target Distortion \( \Delta_2 \).
Decoder 3: Fed Bit streams 1 AND 2. Target Distortion \( \Delta_3 \).

Shannon theoretic study.

Characterize achievable rate-distortion quintuples \((R_1, R_2, \Delta_1, \Delta_2, \Delta_3)\).

Focus on Inner Bound or achievability.
Where and How do the Fixed B-L Codes come in?

Generic Correlation.

$U_1, U_2 \sim \text{Quantized RVs.}$

$\text{Source-Chnl codes.}$
Where and How do the Fixed B-L Codes come in?

$S_1 \quad p_{U_1 | S_1} \quad U_1$

$S_2 \quad p_{U_2 | S_2} \quad U_2$

Generic Correlation.

$U_1, U_2 \sim$ Quantized RVs.

Source-Chnl codes.

$S_1 \quad p_{W} \quad p_{U_1 | W S_1} \quad U_1$

$S_2 \quad p_{U_2 | W S_2} \quad U_2$

Gács-Körner Common part $K$.

Common Quantizer for $K$. 
Where and How do the Fixed B-L Codes come in?

\[ S_1 \xrightarrow{p_{U_1|S_1}} U_1 \]

\[ S_2 \xrightarrow{p_{U_2|S_2}} U_2 \]

**Generic Correlation.**

\[ U_1, U_2 \sim \text{Quantized RVs.} \]

\[ S_1 \xrightarrow{p_{U_1|W S_1}} U_1 \]

\[ S_2 \xrightarrow{p_{U_2|W S_2}} U_2 \]

\[ K \xrightarrow{p_W} W \]

**Gács-Körner Common part** \( K \).

**Common Quantizer for** \( K \).

Common part \( K \) coded via **Identical code** to enable agreement of chosen codeword.

Agreement of chosen codeword enables a larger class of PMFs

\[ U_1 - W S_1 - W S_2 - U_2. \]

Thereby agreement of chosen codeword yields **strict gains**.
Where and How do the Fixed B-L Codes come in?

Common Part

\[ K \]

\[ W \]

\[ S_1 \]

\[ S_2 \]

\[ U_1 \]

\[ U_2 \]

\[ p_W \]

\[ p_{U_1|WS_1} \]

\[ p_{U_2|WS_2} \]

What if \( S_1, S_2 \) only have a 'near GK part'?

How do we ensure Agreement of chosen codeword to obtain gains??
Where and How do the Fixed B-L Codes come in?

Common Part $K$ + identical code guarantees agreement for ANY block-length.

What if $S_1, S_2$ only have a ‘near GK part’?

How do we ensure Agreement of chosen codeword to obtain gains??
Idea of Fixed B-L coding

Suppose $S_1, S_2$ possess **NO GK part**, but $\exists K_1 = f_1(S_1)$ and $K_2 = f_2(S_2)$ with

$$
\begin{array}{c|cc}
\text{p}_{K_1 K_2} & 0 & 1 \\
\hline
0 & 0.495 & 0.005 \\
1 & 0.005 & 0.495 \\
\end{array}
$$

As block-length increases, probability of identical output by the common quantizer decreases.

$$
P(K_n^1 \neq K_n^2) = 1 - P(K_n^1 = K_n^2) = 1 - (P(K_1 = K_2))^n 
\rightarrow 1
$$
Idea of Fixed B-L coding

Suppose $S_1, S_2$ possess **NO GK part**, but $\exists K_1 = f_1(S_1)$ and $K_2 = f_2(S_2)$ with

$$
\begin{array}{ll}
\text{p}_{K_1K_2} & 0 & 1 \\
0 & 0.495 & 0.005 \\
1 & 0.005 & 0.495 \\
\end{array}
$$

$k_1 = 0101001010$

$k_2 = 0101001010$

As block-length increases, prob. of identical output by the common quantizer decreases.

$$
P(K_{\text{nn}} \neq K_{\text{nn}}) = 1 - P(K_{\text{nn}} = K_{\text{nn}}) = 1 - \left( P(K_1 = K_2) \right)_{\text{nn}} = 1 - 0.99^n \to 1$$
Idea of Fixed B-L coding

Suppose $S_1, S_2$ possess NO GK part, but $\exists K_1 = f_1(S_1)$ and $K_2 = f_2(S_2)$ with

\[
\begin{array}{c|c|c}
0 & 1 & K_1 \\
\hline
0 & 0.495 & 0.005 \\
1 & 0.005 & 0.495 \\
\end{array}
\]

As block-length increases, prob. of identical output by the common quantizer decreases.

\[P(K_n^1 \neq K_n^2) = 1 - P(K_n^1 = K_n^2) = 1 - (P(K_1 = K_2))^n \to 1 \text{ as } n \to \infty\]
Idea of Fixed B-L coding

Suppose $S_1, S_2$ possess **NO GK part**, but $\exists K_1 = f_1(S_1)$ and $K_2 = f_2(S_2)$ with

<table>
<thead>
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$p_{K_1K_2}$

As block-length increases, prob. of identical output by the common quantizer decreases.

$$P(K_n^1 \neq K_n^2) = 1 - P(K_n^1 = K_n^2) = 1 - (P(K_1 = K_2))^n \rightarrow 1$$
Idea of Fixed B-L coding

Suppose $S_1, S_2$ possess **NO GK part**, but $\exists K_1 = f_1(S_1)$ and $K_2 = f_2(S_2)$ with

\[
\begin{array}{c|c|c}
p_{K_1K_2} & 0 & 1 \\
0 & 0.495 & 0.005 \\
1 & 0.005 & 0.495 \\
\end{array}
\]

\[
K_1 = 010100101001010101010011101100011011010011010001110101001110110011001010010101011000111010001000101001010101000110101100011101011000111
\]

\[
K_2 = 010100101001000101001010101100011100100101001001101001001001101110001111011001010010010101011000111
\]

As block-length increases, prob. of identical output by the common quantizer decreases.

\[
P(K_1^n \neq K_2^n) = 1 - P(K_1^n = K_2^n) = 1 - (P(K_1 = K_2))^n = 1 - 0.99^n \xrightarrow{n \to \infty} 1
\]
Idea of Fixed B-L coding

\[ S_1 \rightarrow \text{Encoder 1} \]
\[ W_{S_1 S_2} \]
\[ S_2 \rightarrow \text{Encoder 2} \]
Idea of Fixed B-L coding

$01010010100100101010101101100011100101101001010$

$01010010100100101000101101100011100101101001010$

$\text{K}_1^n$

$01010100101101100111010110101010$

$010101001010010010$

$\text{K}_2^n$

$01010010100100101000101101100011100101101001010$

$01010100101101100111010110101010$
Idea of Fixed B-L coding

01010010100100101010101101100011100101101001010
01010010100100101000101101100011100101101001010
01010101010101111000010101010100100
01110100101010101010010101010011110
Common Source Quantizer

0101010101111000101010101001001010
01010101010101111000101010101001010
01110100101010101010010101010011110

Common Source Quantizer

01010010100100101000101101100011100101101001010
01010101010101111000010101010100100
01110100101010101010010101010011110

Common Source Quantizer
Idea of Fixed B-L coding

\[ K_1^n \]

\[ K_2^n \]
Idea of Fixed B-L coding

01010010100

Common Source Quantizer

0111010

0111010

Common Source Quantizer

01010010100

0111010

0111010

01010010100
Idea of Fixed B-L coding

10010101010

Common Source Quantizer

0101010

0111010

Common Source Quantizer

0111010

0111010
Idea of Fixed B-L coding
Idea of Fixed B-L coding

\[00101101001\]

\[10101101\]

\[00101101001\]
Tension in the choice of Block-length

\[ P(K_1^n \neq K_2^n) = 1 - P(K_1^n = K_2^n) = 1 - (P(K_1 = K_2))^n = 1 - 0.99^n \xrightarrow{n \to \infty} 1 \]

Smaller the block-length \( n \), greater the \( P(K_1^n = K_2^n) \).

Larger the block-length \( n \), greater the efficiency of source quantization.

A tension in the choice of block-length of the common shared quantizer.

In the absence of a perfect GK part, a common quantizer of non-asymptotic block-length yields performance gains.
Near GK Part Coded via Common Code of Fixed Block-Length

Conventional Shannon-Theoretic Study: Asymptotically large Block-length chosen as a function of desired probability of error.

Proposed Coding Scheme: Block-length of common Quantizer chosen fixed and non-asymptotic.

Results in non-vanishing prob. of error.

Concatenated Coding Scheme with fixed block-length inner code and an outer code coding over multiple codewords.
Two-layer coding - one fixed B-L, one $\infty$– B-L

B-L $ml$.

0101001010010010101010110110001110010110......1001010

$1 \quad K_1 \quad ml$
Two-layer coding - one fixed B-L, one $\infty - B-L$

B-L $ml$. 

0101001010010010101010110110001110010110……..1001010

1

$K_1$

1

0101001
0100100
1010101
0110110
0011100
1011010

* *

$m$

0101001

1

$l$
Two-layer coding - one fixed B-L, one $\infty$ – B-L

B-L $ml$. 

Common Quantizer

Quantize row-by-row

Fixed B-L $l$
Two-layer coding - one fixed B-L, one $\infty$—B-L

B-L $ml$. 

Layer 2 Quantizer
Block mapping
B-L $ml$

1010111
1110111
0001011
1111111
1101011
1101000
0101001

0101001010010010101010110110001110010110……1001010
1 ml
1010111
1110111
0001011
1111111
1101011
1101000
0101001
Two-layer coding - one fixed B-L, one $\infty$ – B-L

B-L $ml$.

Layer 2 code
Block mapping
B-L $ml$

Layer 1 code
Maps row-by-row
Fixed B-L $l$

Encoding, Commn. and Decoding done Row-Wise. Treated as
Side-information while decoding Layer 2

Conditional Decoding of entire block
Treating decoded layer 1 codewords As side-information
Coding involves Multiple Coding Layers

\[ \begin{align*}
Y_1, K &\rightarrow S, Q_1, Q_2, V, U_1, W_1 \rightarrow S, Q_1, U_1, W_1, V \rightarrow \text{Decoder 1} \rightarrow S, Q_1, U_1 \\
X &\arrow[300]{250}{200}{150}{100}{50}{0} \rightarrow S, Q_1, Q_2, V, U_1, W_1, U_2, W_2 \rightarrow \text{Decoder 2} \rightarrow S, Q_2, U_2 \\
Y_2, K &\rightarrow S, Q_1, Q_2, V, U_2, W_2 \rightarrow S, Q_2, U_2, W_2, V \rightarrow \text{Decoder 3} \rightarrow S, Q_1, Q_2, V, U_1, W_1, U_2, W_2
\end{align*} \]

\( K = \) Gács Körner Common Part. Treated as the source in Multiple Desc. Coding Problem.

\( S, Q_1, Q_2, V \) : The four codes used in MDC to code \( K \).

\( U_1, U_2 \) : Not binned to permit 'stand-alone' reconstruction (in contrast to Berger-Tung problem).

\( W_1, W_2 \) : Berger-Tung codebooks that are binned for compression.

\( S, Q_1, Q_2, V \) : Four Common Code Quantizers. Fixed block-length coding.
A New Achievable rate-distortion region

**Theorem**: \((R_1, R_2, \Delta_1, \Delta_2, \Delta_3)\) is an achievable rate-distortion quintuple if there exists (i) \(l \in \mathbb{N}\), (ii) sets \(K, S, V, U_j, W_j : j \in [2]\), (iii) maps \(f_j : Y_j \rightarrow K\), reconstruction maps \(h_j : U_j \rightarrow Z\), \(g : U_1 \times W_1 \times U_2 \times W_2 \rightarrow Z\), (iv) pmf \(W_{XY}\) defined on \(A := X \times Y \times K \times S \times V \times U \times W\), such that, for \(j \in [2]\)

\[
R_j \geq I(S; K_1) + I(U_j; Y_j | S) + I(W_j; VY_j | U_j, S) - E_l - I(W_j; W_j, U_jV | U_j S),
\]

\[
\Delta_j \geq \mathbb{E}\{d(X, h_j(U_j))\} + \Delta_{\text{max}} \phi_l |A|
\]

\[
R_1 + R_2 \geq I(S; K_1) + I(SV; K_1) - I(W_1; W_2 | U_1U_2SV) + 2E_l
\]

\[
+ \sum_{j=1}^2 [I(U_j; Y_j | S) + I(W_j; V, Y_j | U_j S) - I(W_j; U_j, V | U_j, S)],
\]

\[
\Delta_3 \geq \mathbb{E}\{d(X, g(U, W))\} + \Delta_{\text{max}} \phi_l |A|, \text{ where } \phi_l = \xi^{[l]}(K) + \epsilon_l(p_{K_1SV}) \text{ with } \epsilon_l(p_{K_1SV}) \text{ as given in Prop. 1 [See paper],}
\]

\[
E_l = \epsilon_l(p_{K_1SV}) + 2h_b(\phi_l) + \phi_l \log |A| + 2|A| \phi_l \log(\phi_l^{-1}).
\]