

Optimality of Orthogonal Sequences with Multi-dimensional Signaling

Arun Padakandla

ECE Department
Indian Institute of Science, Bangalore
arunpr@ece.iisc.ernet.in

Rajesh Sundaresan

ECE Department
Indian Institute of Science, Bangalore
rajeshs@ece.iisc.ernet.in

Abstract

The sum capacity on a symbol-synchronous CDMA system having processing gain N and supporting K power constrained users is achieved by employing any set of N orthogonal sequences if a few users are allowed to signal along multiple dimensions. Analogously, the minimum received power (energy-per-chip) on the symbol-synchronous CDMA system supporting K users that demand specified data rates is attained by employing any set of N orthogonal sequences. At most $(N - 1)$ users need to be split and if there are no oversized users, these split users need to signal only in two dimensions each. These results show that sum capacity or minimum sum power can be achieved with minimal downlink signaling.

1. Introduction

Consider a symbol-synchronous code-division multiple access (CDMA) system. The k th user is assigned an N -sequence $s_k \in \mathbb{R}^N$ of unit energy, *i.e.*, $s_k^t s_k = 1$. The processing gain is N chips, and the number of users is K , with $K > N$. User k modulates the vector s_k by its data symbol $X_k \in \mathbb{R}$ and transmits $X_k s_k$ over N chips. This transmission interferes with other users' transmissions and is corrupted by noise. The received signal is modeled by

$$Y = \sum_{k=1}^K s_k X_k + Z,$$

where Z is a zero-mean Gaussian random vector with covariance I_N , the $N \times N$ identity matrix.

We consider two optimization problems. In **Problem I**, user k has a power constraint p_k units per chip, *i.e.*, $\mathbb{E}[X_k^2] \leq N p_k$. The goal is to assign sequences and data rates to users so that the sum of the individual rates at which the users can transmit data reliably (in an asymptotic sense) is maximised. The maximum value C_{sum} is called the sum capacity.

Problem II, a dual to Problem I, is one where user k demands reliable transmission at rate r_k bits/chip or higher. The goal is to assign sequences and powers to users so that despite their mutual interference and noise, each of the users requirement is met, and the sum of the

received powers (energy/chip) at the base-station is minimised.

Viswanath and Anantharam [1] solve Problem I and Guess [2] solves Problem II. Some highlights of their results are the following:

- *Oversized* users, *i.e.*, those capable of transmitting at large powers relative to other users' power constraints in Problem I, and those that demand large rates relative to other users' requirements in Problem II, are best allocated non-interfering sequences.
- Others are allocated generalised Welch-bound equality (GWBE) sequences [1].
- Let $p_{tot} = \sum_{k=1}^K p_k$ and $r_{tot} = \sum_{k=1}^K r_k$. For Problem I, $C_{sum} \leq (1/2) \log(1 + p_{tot})$. For Problem II, the received sum power is lower bounded by $\exp\{2r_{tot}\} - 1$. In both problems, equality holds if and only if no user is oversized.
- No user is oversized if $N x_k \leq x_{tot}$ for every user k where x denotes power in Problem I and rate in Problem II.

Once the optimal sequences are identified and the power or rate allocated to a user determined, they have to be signaled to the typically geographically separated users. When the system is dynamic, either due to the users entering and leaving the system or due to variation of the channel as is typical in wireless communication, the N length sequence vector and a real number representing the power or rate allocated need to be signalled periodically at a significant cost of downlink bandwidth. It is therefore of interest to identify schemes that result in reduced signaling overhead. To some extent this reduced signaling can be achieved by using at most $2N - L - 1$ sequences, as shown in [3], where L is the number of oversized users.

In this paper we show that if some users can be split, N sequences suffice. This is useful because any set of N orthogonal sequences will then work. For example, we may employ the standard basis as in a TDMA system, or a set of Walsh sequences when N is a power of 2 as in a

CDMA system. Moreover, this set can be fixed up front. As the system evolves, it is sufficient to send an index to this set, thereby making the sequence signaling on the downlink rather easy. When there are no oversized users, at most $N - 1$ users are split into exactly two each, and therefore will need two indices. Because their energy is now concentrated in a two-dimensional subspace instead of one, the benefits of spreading (such as robustness to jamming) is obtained to a lesser degree. The allocation is however optimal; it will either maximise sum capacity or minimise sum power.

It is perhaps obvious that N orthogonal sequences suffice if multi-dimensional signaling is allowed. Indeed, the goal of an optimal allocation is to ensure that energy is spread equally in all dimensions leading to a GWBE allocation. A sufficiently fine splitting of the users into virtual users with smaller requirements, coupled with multiple dimensional signaling per user, will achieve this. The interesting aspects of the above result, however, are an identification of the number of users that need be split and the resulting dimensionality of their signaling.

We now discuss some other prior work in the area. The problem of sequence detection has attracted much attention since the work of Rupf and Massey [4] who consider the scenario where users have identical power constraints. Their goal is to maximise the sum capacity. Viswanath and Anantharam [1] extend the results for users with differing power constraints. Guess [2] studies the dual problem (Problem II) and obtains sequences that minimise sum power subject to users' rate constraints. Tropp and others [5] show that sequence design for Problem I is a structured inverse singular value problem [6] and provide a numerically stable algorithm whose complexity is $O(KN)$ floating point operations. The algorithm in [3] has the same $O(KN)$ complexity and numerical stability properties.

Several iterative algorithms for finding optimal sequences have been proposed. As our focus is on finite-step algorithms, we refer the interested reader to a work of Tropp and others [7] and references therein. Sequence design for suboptimal receivers has been studied by Viswanath and others [8] and Guess [9]. In our work, we place no restriction on the receivers.

The paper is organised as follows. Section 2 provides the preliminaries and states the problems. Section 3 describes the sequence allocation algorithms. In Section 4 we propose the technique of multi-dimensional signaling and provide the proof for sufficiency of N orthogonal sequences. Section 5 provides some concluding remarks.

2. Preliminaries and Problem Statements

Suppose user k is assigned sequence s_k and is received at power p_k . Let S be the $N \times K$ matrix $[s_1 \ s_2 \ \dots \ s_K]$.

Then, the capacity region [10] can be written as

$$C(S, p) = \bigcap_{J \subset \{1, \dots, K\}} \left\{ (r_1, \dots, r_K) \in \mathbb{R}_+^K : \sum_{k \in J} r_k \leq \frac{1}{2N} \log \left| I_N + \sum_{k \in J} N p_k \cdot s_k s_k^t \right| \right\}, \quad (1)$$

where $|A|$ denotes the determinant of the matrix A , and r_k is user k 's data rate in bits/chip.

The rate vector $r = (r_1, \dots, r_K)$ is a *vertex* of this capacity region if

$$r_k \triangleq \frac{1}{2N} \log |A_k| - \frac{1}{2N} \log |A_{k-1}|, \quad 1 \leq k \leq K, \quad (2)$$

where $A_0 \triangleq I_N$, and $A_k \triangleq (A_{k-1} + N p_k \cdot s_k s_k^t)$ for $k = 1, \dots, K$. The matrix A_k is real, symmetric, and positive-definite.

The vertex (see (2)) satisfies $r_k \geq 0$ and $r \in C(S, p)$. These are deduced as follows. Let $\sigma(A_k) = (\lambda_1^{(k)}, \dots, \lambda_N^{(k)})$. Clearly $\lambda_1^{(0)} = \dots = \lambda_N^{(0)} = 1$. A well-known result due to Weyl (see for example [11, Section 4.3]) indicates

$$\begin{aligned} \lambda_1^{(k)} &\geq \lambda_1^{(k-1)} \geq \lambda_2^{(k)} \geq \lambda_2^{(k-1)} \geq \dots \\ &\dots \geq \lambda_N^{(k)} \geq \lambda_N^{(k-1)} \geq 1, \end{aligned}$$

which implies that $|A_k| \geq |A_{k-1}|$, and hence $r_k \geq 0$. Moreover, $r \in C(S, p)$ because this rate point can be achieved via successive decoding. (Alternatively, $C(S, p)$ is a polymatroid [12], and therefore contains all its vertices in \mathbb{R}_+^N).

With the above ideas fixed, let us now re-state Problems I and II.

- **Problem I** : Given a per user power constraint of $p = (p_1, \dots, p_K)$ where no user is oversized (*i.e.*, $N p_k \leq p_{tot}$ for every user k), find S and r that satisfy $r \in C(S, p)$ and $r_{tot} = (1/2) \log(1 + p_{tot})$, the maximum sum-rate among all sequence and rate allocations.
- **Problem II** : Given a per user rate requirement of $r = (r_1, \dots, r_K)$ bits/chip where no user is oversized (*i.e.*, $N r_k \leq r_{tot}$ for every user k), find S and p that satisfy $r \in C(S, p)$ and $p_{tot} = \exp\{2r_{tot}\} - 1$, the minimum value among all power and sequence allocations.

In the following section we describe the sequence design algorithm that solve the above problems. Those algorithms will be our starting points to show that N orthogonal sequences are sufficient to achieve maximum sum capacity or minimum sum power.

3. Sequence Assignments

In this section, we state the algorithms for sequence allocation for Problems I and II. The proof of correctness is provided in [13]. The algorithms assign sequences to users one after another. The order of the users is immaterial. We therefore assume that users are assigned in the increasing order of their indices. We discuss the intuition behind the algorithms.

Allocating powers to various dimensions can be thought of as pouring water in N columns, each column representing a dimension. For systems with no oversized users, the water-filling solution is optimal. Thus we need to get equal powers in all dimensions to attain optimality. The algorithm adds one user after another ensuring that the added user's requirements are met at every stage. This is done by pouring power in an already partially filled dimension before going to an empty dimension. Occasionally, a user's requirement will not fit in a given dimension, say n . In such cases, the user fills up dimension n to the maximum value suggested by the water-filling solution, and pours the remaining power in the empty dimension $n + 1$. If there are no oversized users, it is possible to allocate a single sequence to such a user, as shown in [13].

3.1. Algorithm for Problem II

Let r_k denote the rate requirement of user k . The algorithms make use of a subroutine $c(A, \hat{\lambda})$ where $\sigma(A) = \lambda = (\lambda_1, \dots, \lambda_N)$ and $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_N)$ interlace, i.e.,

$$\hat{\lambda}_1 \geq \lambda_1 \geq \hat{\lambda}_2 \geq \lambda_2 \geq \dots \geq \hat{\lambda}_N \geq \lambda_N. \quad (3)$$

The output vector c is such that $\sigma(A + cc^*) = \hat{\lambda}$. For a proof of this fact, and for a proof of the correctness of the following algorithms, see [13]. The proofs are not central to the contribution in this paper.

Algorithm 1 Problem II

- **Initialisation:** Set $\lambda_n^{(k)} \leftarrow 1$ for $k = 0, 1, \dots, K$ and $n = 1, \dots, N$. Set the user index $k \leftarrow 1$, dimension $n \leftarrow 1$, $\lambda_{\max} \leftarrow \exp\{2r_{\text{tot}}\}$, and $A_0 \leftarrow I_N$.
- **Step 1:** If $k > K$, stop.
- **Step 2 (a):** If $\lambda_n^{(k-1)} \cdot \exp\{2Nr_k\} < \lambda_{\max}$, then set $\lambda_n^{(k)} \leftarrow \lambda_n^{(k-1)} \cdot \exp\{2Nr_k\}$ and go to **Step 3**.
- **Step 2 (b):** If $\lambda_n^{(k-1)} \cdot \exp\{2Nr_k\} = \lambda_{\max}$, then set $\lambda_n^{(j)} \leftarrow \lambda_{\max}$ for $j = k, \dots, K$. Set $n \leftarrow n + 1$ and go to **Step 3**.
- **Step 2 (c):** If $\lambda_n^{(k-1)} \cdot \exp\{2Nr_k\} > \lambda_{\max}$, then set $\lambda_n^{(j)} \leftarrow \lambda_{\max}$ for $j = k, \dots, K$, and $\lambda_{n+1}^{(k)} \leftarrow \lambda_n^{(k-1)} \cdot \exp\{2Nr_k\} / \lambda_{\max}$. Also set $n \leftarrow n + 1$.

- **Step 3:** Identify the vector $c_k = c(A_{k-1}, \lambda^{(k)})$. Then set $s_k \leftarrow c_k / \|c_k\|$, $p_k \leftarrow (c_k^t c_k) / N$. This provides the sequence and power for user k . Finally, set $A_k \leftarrow A_{k-1} + c_k c_k^t$, $k \leftarrow k + 1$, and go to **Step 1**.

3.2. Algorithm for Problem I

We will now describe the algorithm for the dual problem of sequence and rate allocation for sum capacity maximisation given users' power constraints $p = (p_1, \dots, p_K)$.

Algorithm 2 Problem I

- **Initialisation:** Set $\lambda_n^{(k)} \leftarrow 1$ for $k = 0, 1, \dots, K$ and $n = 1, \dots, N$. Set the user index $k \leftarrow 1$, dimension $n \leftarrow 1$, $\lambda_{\max} \leftarrow 1 + p_{\text{tot}}$, and $A_0 \leftarrow I_N$.
- **Step 1:** If $k > K$, stop.
- **Step 2 (a):** If $\lambda_n^{(k-1)} + Np_k < \lambda_{\max}$, then set $\lambda_n^{(k)} \leftarrow \lambda_n^{(k-1)} + Np_k$ and go to **Step 3**.
- **Step 2 (b):** If $\lambda_n^{(k-1)} + Np_k = \lambda_{\max}$, then set $\lambda_n^{(j)} \leftarrow \lambda_{\max}$ for $j = k, \dots, K$. Also set $n \leftarrow n + 1$ and go to **Step 3**.
- **Step 2 (c):** If $\lambda_n^{(k-1)} + Np_k > \lambda_{\max}$, then set $\lambda_n^{(j)} \leftarrow \lambda_{\max}$ for $j = k, \dots, K$, and $\lambda_{n+1}^{(k)} \leftarrow 1 + \lambda_n^{(k-1)} + Np_k - \lambda_{\max}$. Also set $n \leftarrow n + 1$.
- **Step 3:** Identify the vector $c_k = c(A_{k-1}, \lambda^{(k)})$. Then set $s_k \leftarrow c_k / \|c_k\|$, $r_k \leftarrow \frac{1}{2N} \log |A_k| - \frac{1}{2N} \log |A_{k-1}|$. This provides the sequence and rate for user k . Finally, set $A_k \leftarrow A_{k-1} + c_k c_k^t$, $k \leftarrow k + 1$, and go to **Step 1**.

4. Sufficiency of N orthogonal sequences

We now show how to achieve sum capacity (respectively, minimum sum power) by using at most N orthogonal sequences. The above algorithms confine each user to signal along a single dimension. It is possible in some cases that the algorithms in Section 3 lead to a set of orthogonal sequences. The following example illustrates this.

Example 3 Let $N = 3$ and $K = 5$. Let the five users have rate requirements $r = (r_1, r_2, r_3, r_4, r_5) = (2, 2, 4, 3, 1)$ bits/chip. Allocate powers and sequences to these users so that their rate requirements are met, and the sum power minimised.

No user is oversized in Example 3. Assigning sequences in the increasing order of their indices, Algorithm 1 results in the following sequence assignment. Users 1 and 2 share the sequence $(1, 0, 0)^t$, user 3 is assigned $(0, 1, 0)^t$, and users 4, 5 share $(0, 0, 1)^t$. The key to attaining orthogonality is the fact that after each user is added, exactly one eigenvalue changes. Users 1 and 2 exactly fill

up dimension 1, user 3 fills up dimension 2, and users 4 and 5 fill up dimension 3. This motivates the following definition.

Definition 4 A vector $x = (x_1, x_2, \dots, x_K)$ has a symmetric sum partition of size N , if there is a partition of the users $\{1, 2, \dots, K\}$ into N subsets S_1, S_2, \dots, S_N , such that

$$\sum_{k \in S_n} x_k = \frac{1}{N} \sum_{k=1}^K x_k = \frac{x_{tot}}{N}, \quad (4)$$

for $n=1, 2, \dots, N$. The subsets S_1, S_2, \dots, S_N will be referred to as the symmetric sum partition.

Remark : If x has a symmetric sum partition of size N , no user is oversized. This is because, any user k' belongs to S_n , for some n , and (4) implies $x_{k'} \leq \sum_{k \in S_n} x_k = x_{tot}/N$. Note that the rate constraint vector r of Example 3 has a symmetric sum partition of size 3. The partition is $\{1, 2\}, \{3\}$, and $\{4, 5\}$.

Proposition 5 If the rate vector r has a symmetric sum partition of size N , then N orthogonal sequences are sufficient to attain the minimum sum power $p_{tot} = \exp\{2r_{tot}\} - 1$. Analogously, if the power constraint vector p has a symmetric sum partition of size N , then N orthogonal sequences are sufficient to attain the sum capacity $r_{tot} = \frac{1}{2} \log(1 + p_{tot})$. \square

Proof: We will prove the proposition for Problem II. A similar argument holds for Problem I.

Let S_1, S_2, \dots, S_N be the symmetric sum partition of the users. Assign sequences and powers as follows: if $k \in S_n$, then

$$s_k = e_n, \quad (5)$$

$$p_k = \frac{\mathfrak{J}_k(n)}{N} [\exp\{2Nr_k\} - 1], \quad (6)$$

where

$$\mathfrak{J}_k(n) \triangleq \exp \left\{ 2N \sum_{j: j \in S_n, j < k} r_j \right\} \quad (7)$$

is the interference suffered by user k due to presence of other users in the same subset S_n with a smaller index.

It is easy to see that

$$r_k = \frac{1}{2N} \log \left(1 + \frac{Np_k}{\mathfrak{J}_k(n)} \right) \quad (8)$$

is achievable via successive interference cancelation, where the highest index user in this subset is decoded first. Users in other subsets do not cause interference to users in this subset. Observe that the total power assigned to users in any subset S_n is given by

$$\sum_{k \in S_n} p_k = \frac{1}{N} (\exp\{2r_{tot}\} - 1), \quad (9)$$

where (9) follows by substitution of (6) and (7) in the left side of (9) and by observing that the resulting sum over S_n has only two terms that survive.

From (9) the total power allocated to all users is $p_{tot} = \sum_{n=1}^N \sum_{k \in S_n} p_k = \exp\{2r_{tot}\} - 1$, thus showing that N orthogonal sequences are optimal. \blacksquare

Checking for the existence of a symmetric sum partition is as hard as checking to see if there is a subset with a certain target sum. This is known to be an NP-complete problem when the input constraints are integers. However, we show how to manufacture a symmetric sum partition of size N from a given set of power constraints or rate requirements.

Proposition 6 Every strictly positive vector x representing power constraints or rate requirements for K non-oversized users can be cast into a vector x' for K' virtual users, where $K \leq K' \leq K + N - 1$, and x' is such that it has a symmetric sum partition of size N . Moreover x' is obtained by splitting $K' - K$ users into exactly two virtual users each. \square

Proof: The cumulative requirement $X_k \triangleq \sum_{i \leq k} x_i$ is a strictly increasing function of k satisfying $X_K = x_{tot}$. Hence there exist $N - 1$ distinct users with indices $k_j, j = 1, 2, \dots, N - 1$, such that

$$\begin{aligned} X_{k_{j-1}} &= \sum_{i=1}^{k_{j-1}} x_i < \frac{jx_{tot}}{N}, \\ X_{k_j} &= \sum_{i=1}^{k_j} x_i \geq \frac{jx_{tot}}{N}. \end{aligned} \quad (10)$$

If strict inequality holds in (10), split user k_j 's rate as

$$x_{k_j} = \underbrace{\left(\frac{jx_{tot}}{N} - X_{k_{j-1}} \right)}_{x'_{k_j}} + \underbrace{\left(X_{k_j} - \frac{jx_{tot}}{N} \right)}_{x''_{k_j}}. \quad (11)$$

If equality holds in (10) leave the user as is. For users that will be split, obtain x' from x by replacing x_{k_j} , the requirement for user k_j , by requirements x'_{k_j} and x''_{k_j} for two virtual users, where x'_{k_j} and x''_{k_j} are as in (11). Clearly, x' is a vector of size K' , $K \leq K' \leq K + N - 1$, and x' has a symmetric sum partition of size N . \blacksquare

Users whose rates or power constraints are split are assigned two orthogonal sequences. The design subsequently results in N orthogonal sequences.

It is immediate that even for the case with oversized users, the oversized users will be split into at most N virtual users. Non-oversized users will be split into at most two virtual users. Thus N orthogonal sequences are sufficient if rate or power splitting and multi-dimensional signaling is allowed for some users. We make this precise in the following Proposition. The proof is identical to the proof of Proposition 6 and therefore omitted.

Proposition 7 Every strictly positive vector x representing power constraints or rate requirements for K users, regardless of the presence of oversized users, can be cast into a vector x' of size K' , where x' has a symmetric sum partition of size N , and $K \leq K' \leq K + N - 1$. \square

5. Concluding Remarks

The work in this paper was primarily motivated by a desire to reduce the amount of signaling necessary to communicate the sequences to the geographically separated users. We saw that with a small penalty in the spreading factor for a few users, N orthogonal sequences are sufficient. The users and the base station can then agree on a fixed orthogonal set of N sequences. The base station only needs to signal the powers/rates and the indices corresponding to the sequences allocated to a user.

The assumed multiple access channel model, however, has severe limitations. The uplink wireless channel typically suffers from multipath effects, fading, and asynchronism. Moreover, the users are not active all the time. Yet, if the users can all be synchronised via, for example, a Global Positioning System (GPS) receiver, our results give some interesting design insights. For frequency-flat slow fading channels where all the users with a tight delay constraint have to be served simultaneously, multi-dimensional signaling (more commonly referred to multi-code where the code is a spreading sequence) can allow communication at sum capacity or minimum power. Orthogonal sequences are sufficient and the signaling in the downlink is significantly reduced. A successively canceling decoder is necessary, but the complexity of this receiver is reduced to a great extent because optimal decoding for the K users decouples into decoding for N separate non-interfering groups.

Fairness of the allocation has not been considered in this paper. However, with the splitting approach that separates virtual users into groups, the decoding order of virtual users within a group can be cycled to get a fairer allocation. The first user to be decoded in a group treats all others in the group as interference and suffers the most. Cycling ensures that this and other such disadvantageous positions are shared in time by all users.

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