COSC 317 Worksheet 3

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3. Distributive Lattices and Boolean Algebras

3.1. More on Lattices.

DEFINITION 3.1 (isomorphic lattices). Two lattices, $(L, \wedge, \vee)$ and $(L', \wedge', \vee')$, are *isomorphic* iff there is a one-to-one relation between the elements of L and L' that makes their meet and join operations correspond as well. *This is an informal definition. Turn it into a formal definition.*

THEOREM 3.1. $(X, \diamond)$ is a semilattice iff (1) $x \diamond x = x$ and (2) $(x \diamond y) \diamond z = z \diamond (x \diamond y)$, for all $x, y, z \in X$.

THEOREM 3.2. There are exactly four nonempty, nonisomorphic lattices with four or less elements.

PROBLEM 3.1. Draw Hasse diagrams of five nonisomorphic lattices of five elements each.

THEOREM 3.3. Three of the nonisomorphic lattices of five elements are self-dual.

THEOREM 3.4. Every lattice of five elements is isomorphic to one of lattices from Prob. 3.1. *Hint:* Add $\top$ and $\bot$ to each possible 3-element poset.

THEOREM 3.5. Show that if a poset of five or fewer elements includes $\top$ and $\bot$, then the poset is a lattice.

THEOREM 3.6. Suppose that $(X, \wedge, \vee)$ is an algebra with two binary operators $\wedge, \vee$ that satisfy L2–L4 (Def. 2.4). Show that these identities imply L1 and therefore that X is a lattice.

3.2. Distributive Lattices.

THEOREM 3.7. The following two propositions are equivalent:

L6′: $\forall x, y, z : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
L6″: $\forall x, y, z : x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In other words, this theorem states that the first equality holds for all elements if and only if the second equality holds for all elements.

DEFINITION 3.2 (distributive lattice). A lattice is *distributive* if equality L6′ (or, equivalently, L6″) holds.

PROBLEM 3.2. Determine which of the lattices in Exs. 1.1–1.11 are distributive, or the conditions under which they would be distributive.

THEOREM 3.8. In a distributive lattice, if for some c, $c \wedge x = c \wedge y$ and $c \vee x = c \vee y$, then $x = y$.

DEFINITION 3.3 (modular lattice). A lattice is *modular* if it satisfies:
  L5: If $x \sqsubseteq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

THEOREM 3.9. Every distributive lattice is modular.

THEOREM 3.10. Not every lattice is modular.

THEOREM 3.11. Not every modular lattice is distributive.

THEOREM 3.12. L1, L2, L3, and L6 imply $x \vee (x \wedge y) = x \wedge (x \vee y)$.

THEOREM 3.13. If L is a distributive lattice, and we add to it new elements $\bot$ and $\top$ satisfying $\bot \sqsubset x \sqsubset \top$ for all $x \in L$, then the result will also be a distributive lattice.

THEOREM 3.14. Modularity (L5) is equivalent to: if $x \sqsubset z$, then $x \vee (y \wedge z) \not\succ (x \vee y) \wedge z$.

DEFINITION 3.4 (complement). A *complement* of an element x of a complete lattice is an element y satisfying $x \wedge y = \bot$ and $x \vee y = \top$.

THEOREM 3.15. If an element of a distributed lattice has a complement, then it is unique. *Hint:* Use Thm. 3.9.

The unique complement of x is written x'. Thus,
  L8: $x \wedge x' = \bot$ and $x \vee x' = \top$.

THEOREM 3.16. In a complete distributive lattice, the set of all complemented elements is a sublattice.

PROBLEM 3.3. Give an example of a nondistributive modular lattice of seven elements in which the complemented elements do *not* form a sublattice.

DEFINITION 3.5 (complemented lattice). A *complemented lattice* is a (nonempty) complete lattice in which every element is complemented.

THEOREM 3.17. Any lattice is a sublattice of a complemented lattice with just three additional elements. If the lattice is finite, then just one additional element is required.

DEFINITION 3.6 (Boolean lattice). A *Boolean lattice* is a complemented distributive lattice.

THEOREM 3.18. In a Boolean lattice:

L9 (Involution): $(x')' = x$.
L10 (DeMorgan): $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$.

THEOREM 3.19. Every Boolean lattice is self-dual.

DEFINITION 3.7 (Boolean algebra). A *Boolean algebra* is a set containing $\top, \bot$ with two binary operations $(\wedge, \vee)$ and one unary operation $(')$ satisfying L1–L10.