Review of Gaussian (Normal) Distributions

Bruce J. MacLennan

September 4, 2002

1 Density Functions

Standard Gaussian (Normal) Density Function:

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

(General) Gaussian (Normal) Density Function:

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Bernoulli Trials: n binary trials with probabilities $p + q = 1$. For large n and np, well approximated by a Gaussian with $\mu = np$, $\sigma^2 = npq$.

$$\left(\begin{array}{c} x \\ n \end{array}\right) p^x q^{n-x} \approx f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi npq}}\exp\left[-\frac{(x-np)^2}{2npq}\right]$$

2 Distribution Functions

Standard Normal Distribution Function:

$$P(a,b) = \int_a^b \phi(t)\mathrm{d}t = \Pr\{a \le t \le b\}$$

General Gaussian Distribution Function:

$$\Pr\{x_1 \le x \le x_2\} = P\left(\frac{x_1-\mu}{\sigma}, \frac{x_2-\mu}{\sigma}\right)$$

Areas Relative to Standard Deviation:

$$\begin{aligned} P(0,1) &= 34\% \\ P(1,2) &= 14\% \\ P(2,3) &= 2\% \\ P(3,\infty) &= 0.13\% \end{aligned}$$

3 Cumulative Distributions

Beware differences in notation and terminology!

Probability Integral:

$$\Phi(x) = P(-\infty, x) = \int_{-\infty}^{x} \phi(t)\mathrm{d}t$$

Note $P(0,x) = \Phi(x) - \frac{1}{2}$. $\Phi(1) = 84\%$, $\Phi(2) = 98\%$, $\Phi(3) = 99.9\%$.

For arbitrary μ, σ:

$$\begin{aligned}
\Pr\{t \leq t_c\} &= \Phi\left(\frac{t_c - \mu}{\sigma}\right) \\
\Pr\{t \geq t_c\} &= 1 - \Phi\left(\frac{t_c - \mu}{\sigma}\right) \\
\Pr\{a \leq t \leq b\} &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) \\
\Pr\{|t| \geq t_c\} &= 2 - 2\Phi(t_c/\sigma), \quad \text{for } \mu = 0 \\
\Pr\{|t| \leq t_c\} &= 2\Phi(t_c/\sigma) - 1, \quad \text{for } \mu = 0
\end{aligned}$$

Error Function:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$$

Complementary Error Function:

$$\operatorname{erfc} x = 1 - \operatorname{erf} x$$

Conversions:

$$\begin{aligned}
\Phi(x) &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \\
\operatorname{erf} x &= 2\Phi(\sqrt{2}x) - 1 \\
&= \Pr\{|t| \leq \sqrt{2}x\} \\
P(0,x) &= \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \\
\operatorname{erf} x &= 2P(0, \sqrt{2}x)
\end{aligned}$$

Approximations:

$$\begin{aligned}
\operatorname{erf} x &\approx \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots\right), \quad \text{for } |x| \ll 1 \\
\operatorname{erfc} x &\approx \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{3}{4x^4} - \cdots\right), \quad \text{for } |x| \gg 1
\end{aligned}$$