Lecture 22

Reading

- Flake, ch. 20 ("Genetics and Evolution")

Imprinting Multiple Patterns

- Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p$ be patterns to be imprinted
- Define the sum-of-outer-products matrix:

$$W_{ij} = \tfrac{1}{n} \sum_{k=1}^{p} x_i^k x_j^k$$

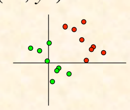
$$\mathbf{W} = \tfrac{1}{n} \sum_{k=1}^{p} \mathbf{x}^k \left(\mathbf{x}^k\right)^{\mathrm{T}}$$

Definition of Covariance

Consider samples $(x^1, y^1), (x^2, y^2), \ldots, (x^N, y^N)$

Let $\bar{x} = \langle x^k \rangle$ and $\bar{y} = \langle y^k \rangle$

Covariance of x and y values:



$$C_{xy} = \left\langle \left(x^k - \bar{x}\right)\left(y^k - \bar{y}\right) \right\rangle$$
$$= \left\langle x^k y^k - \bar{x} y^k - x^k \bar{y} + \bar{x} \cdot \bar{y} \right\rangle$$
$$= \left\langle x^k y^k \right\rangle - \bar{x}\left\langle y^k \right\rangle - \left\langle x^k \right\rangle \bar{y} + \bar{x} \cdot \bar{y}$$
$$= \left\langle x^k y^k \right\rangle - \bar{x} \cdot \bar{y} - \bar{x} \cdot \bar{y} + \bar{x} \cdot \bar{y}$$
$$C_{xy} = \left\langle x^k y^k \right\rangle - \bar{x} \cdot \bar{y}$$

Weights & the Covariance Matrix

Sample pattern vectors: $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p$

Covariance of i^{th} and j^{th} components:

$$C_{ij} = \left\langle x_i^k x_j^k \right\rangle - \overline{x_i} \cdot \overline{x_j}$$

If $\forall i : \overline{x_i} = 0$ (±1 equally likely in all positions):

$$C_{ij} = \left\langle x_i^k x_j^k \right\rangle = \tfrac{1}{p} \sum_{k=1}^{p} x_i^k y_j^k$$

$$\therefore \mathbf{W} = \tfrac{p}{n} \mathbf{C}$$

Characteristics of Hopfield Memory

- Distributed ("holographic")
  - every pattern is stored in every location (weight)
- Robust
  - correct retrieval in spite of noise or error in patterns
  - correct operation in spite of considerable weight damage or noise

Stability of Imprinted Memories

- Suppose the state is one of the imprinted patterns $\mathbf{x}^m$
- Then: $\mathbf{h} = \mathbf{W}\mathbf{x}^m = \left[\frac{1}{n}\sum_k \mathbf{x}^k \left(\mathbf{x}^k\right)^{\mathrm{T}}\right]\mathbf{x}^m$

$$= \frac{1}{n}\sum_k \mathbf{x}^k \left(\mathbf{x}^k\right)^{\mathrm{T}} \mathbf{x}^m$$

$$= \frac{1}{n}\mathbf{x}^m \left(\mathbf{x}^m\right)^{\mathrm{T}} \mathbf{x}^m + \frac{1}{n}\sum_{k \neq m} \mathbf{x}^k \left(\mathbf{x}^k\right)^{\mathrm{T}} \mathbf{x}^m$$

$$= \mathbf{x}^m + \frac{1}{n}\sum_{k \neq m} \left(\mathbf{x}^k \cdot \mathbf{x}^m\right)\mathbf{x}^k$$

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Interpretation of Inner Products

- $\mathbf{x}^k \cdot \mathbf{x}^m = n$ if they are identical 
  - highly correlated
- $\mathbf{x}^k \cdot \mathbf{x}^m = -n$ if they are complementary 
  - highly correlated (reversed)
- $\mathbf{x}^k \cdot \mathbf{x}^m = 0$ if they are orthogonal 
  - largely uncorrelated
- $\mathbf{x}^k \cdot \mathbf{x}^m$ measures the *crosstalk* between patterns k and m 

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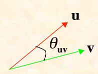
Cosines and Inner products

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta_{\mathbf{uv}}$$



If $\mathbf{u}$ is bipolar, then $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = n$

Hence, $\mathbf{u} \cdot \mathbf{v} = \sqrt{n}\sqrt{n}\cos\theta_{\mathbf{uv}} = n\cos\theta_{\mathbf{uv}}$

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Conditions for Stability

Stability of entire pattern:

$$\mathbf{x}^m = \operatorname{sgn}\left(\mathbf{x}^m + \frac{1}{n}\sum_{k \neq m} \mathbf{x}^k \cos\theta_{km}\right)$$

Stability of a single bit:

$$x_i^m = \operatorname{sgn}\left(x_i^m + \frac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km}\right)$$

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Sufficient Conditions for Instability (Case 1)

Suppose $x_i^m = -1$. Then unstable if:

$$(-1) + \frac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} > 0$$

$$\frac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} > 1$$

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Sufficient Conditions for Instability (Case 2)

Suppose $x_i^m = +1$. Then unstable if:

$$(+1) + \frac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} < 0$$

$$\frac{1}{n}\sum_{k \neq m} x_i^k \cos\theta_{km} < -1$$

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Sufficient Conditions for Stability

$$\left| \frac{1}{n} \sum_{k \neq m} x_i^k \cos\theta_{km} \right| \leq 1$$

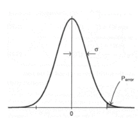
The crosstalk with the sought pattern must be sufficiently small

Capacity of Hopfield Memory

- Depends on the patterns imprinted
- If orthogonal, $p_{\max} = n$
  - but every state is stable ⇒ trivial basins
- So $p_{\max} < n$
- Let load parameter $\alpha = p / n$

Single Bit Stability Analysis

- For simplicity, suppose $\mathbf{x}^k$ are random
- Then $\mathbf{x}^k \cdot \mathbf{x}^m$ are sums of n random ±1
  - binomial distribution ≈ Gaussian
  - in range $-n, \ldots, +n$
  - with mean $\mu = 0$
  - and variance $\sigma^2 = n$
- Probability sum $> t$:



$$\tfrac{1}{2}\left[1 - \operatorname{erf}\left(\frac{t}{\sqrt{2n}}\right)\right]$$

[See "Review of Gaussian (Normal) Distributions" on course website]

Approximation of Probability

Let crosstalk $C_i^m = \frac{1}{n} \sum_{k \neq m} x_i^k \left(\mathbf{x}^k \cdot \mathbf{x}^m\right)$

We want $\Pr\left\{C_i^m > 1\right\} = \Pr\left\{nC_i^m > n\right\}$

Note: $nC_i^m = \sum_{\substack{k=1 \\ k \neq m}}^{p} \sum_{j=1}^{n} x_i^k x_j^k x_j^m$

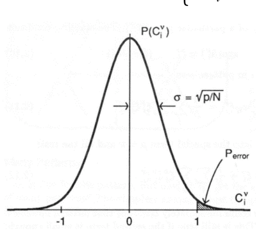
A sum of $n(p-1) \approx np$ random ±1s

Variance $\sigma^2 = np$

Probability of Bit Instability

$$\Pr\left\{nC_i^m > n\right\} = \tfrac{1}{2}\left[1 - \operatorname{erf}\left(\frac{n}{\sqrt{2np}}\right)\right]$$

$$= \tfrac{1}{2}\left[1 - \operatorname{erf}\left(\sqrt{n/2p}\right)\right]$$



(fig. from Hertz & al. *Intr. Theory Neur. Comp.*)

Tabulated Probability of Single-Bit Instability

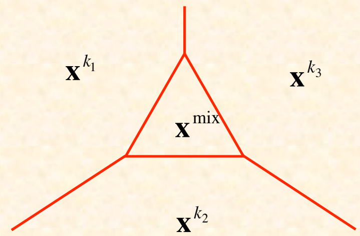
P_{error}	α
0.1%	0.105
0.36%	0.138
1%	0.185
5%	0.37
10%	0.61

(table from Hertz & al. *Intr. Theory Neur. Comp.*)

Spurious Attractors

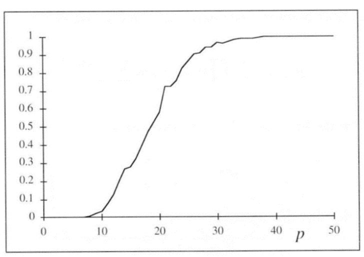
- Mixture states:
  - sums or differences of odd numbers of retrieval states
  - number increases combinatorially with *p*
  - shallower, smaller basins
  - basins of mixtures swamp basins of retrieval states ⇒ overload
  - useful as combinatorial generalizations?
  - self-coupling generates spurious attractors
- Spin-glass states:
  - not correlated with any finite number of imprinted patterns
  - occur beyond overload because weights effectively random

Basins of Mixture States



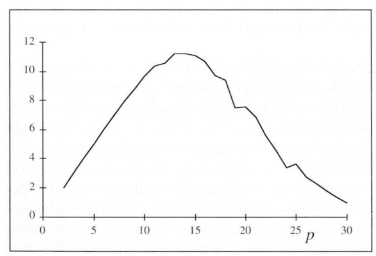
$$x_i^{\text{mix}} = \text{sgn}\left(x_i^{k_1} + x_i^{k_2} + x_i^{k_3}\right)$$

Fraction of Unstable Imprints ($n = 100$)



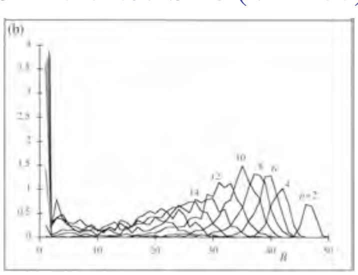
 (fig from Bar-Yam)

Number of Stable Imprints ($n = 100$)



 (fig from Bar-Yam)

Number of Imprints with Basins of Indicated Size ($n = 100$)



 (fig from Bar-Yam)

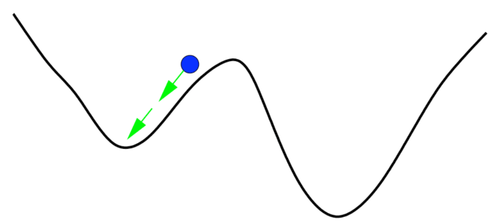
Summary of Capacity Results

- Absolute limit: $p_{\text{max}} < \alpha_c n = 0.138\ n$
- If a small number of errors in each pattern permitted: $p_{\text{max}} \propto n$
- If all or most patterns must be recalled perfectly: $p_{\text{max}} \propto n / \log n$
- Recall: all this analysis is based on *random* patterns
- Unrealistic, but sometimes can be arranged

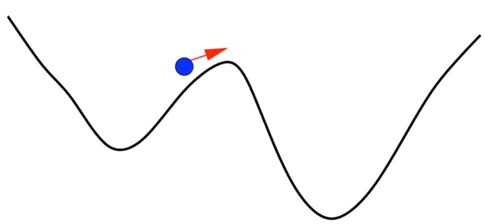
Stochastic Neural Networks

(in particular, the stochastic Hopfield network)

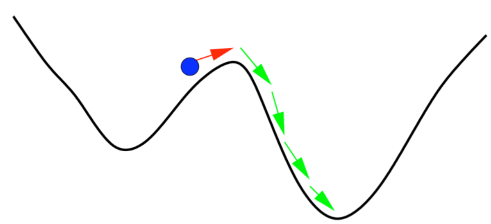
Trapping in Local Minimum



Escape from Local Minimum



Escape from Local Minimum



Motivation

- Idea: with low probability, go against the local field
  - move up the energy surface
  - make the “wrong” microdecision
- Potential value for optimization: escape from local optima
- Potential value for associative memory: escape from spurious states
  - because they have higher energy than imprinted states

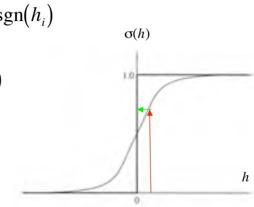
The Stochastic Neuron

Deterministic neuron: $s_i' = \operatorname{sgn}(h_i)$

$$\Pr\{s_i' = +1\} = \Theta(h_i)$$
$$\Pr\{s_i' = -1\} = 1 - \Theta(h_i)$$

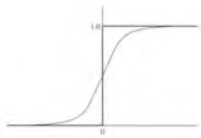
Stochastic neuron:

$$\Pr\{s_i' = +1\} = \sigma(h_i)$$
$$\Pr\{s_i' = -1\} = 1 - \sigma(h_i)$$



Logistic sigmoid: $\sigma(h) = \dfrac{1}{1 + \exp(-2h/T)}$

Properties of Logistic Sigmoid



$$\sigma(h) = \frac{1}{1 + e^{-2h/T}}$$

- As $h \to +\infty$, $\sigma(h) \to 1$
- As $h \to -\infty$, $\sigma(h) \to 0$
- $\sigma(0) = 1/2$

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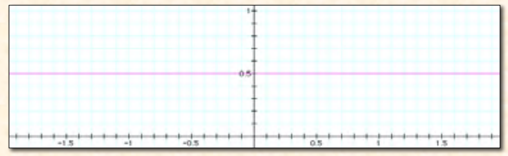
Logistic Sigmoid With Varying *T*



T varying from 0.05 to ∞ ($1/T = \beta = 0, 1, 2, \ldots, 20$)

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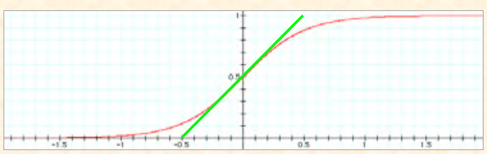
Logistic Sigmoid $T = 0.5$



Slope at origin = $1 / 2T$

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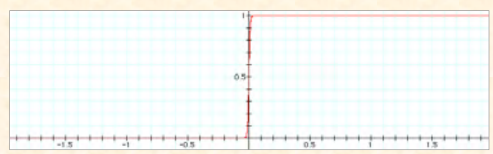
Logistic Sigmoid $T = 0.01$



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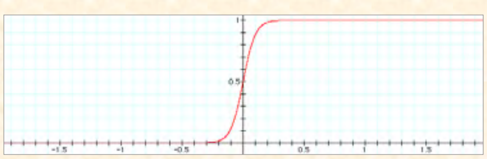
Logistic Sigmoid $T = 0.1$



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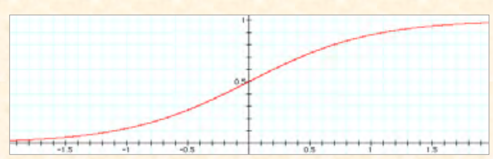
Logistic Sigmoid $T = 1$



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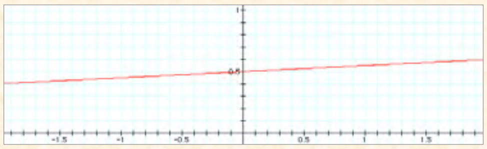
Logistic Sigmoid
$T = 10$



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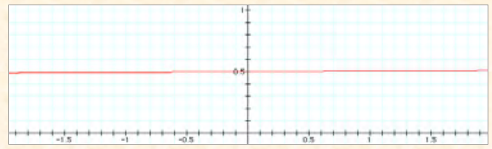
Logistic Sigmoid
$T = 100$



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Pseudo-Temperature

- Temperature = measure of thermal energy (heat)
- Thermal energy = vibrational energy of molecules
- A source of random motion
- Pseudo-temperature = a measure of nondirected (random) change
- Logistic sigmoid gives same equilibrium probabilities as Boltzmann-Gibbs distribution

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Transition Probability

Recall, change in energy $\Delta E = -\Delta s_k h_k$

$$= 2 s_k h_k$$

$$\Pr\{s'_k = \pm 1 | s_k = \mp 1\} = \sigma(\pm h_k) = \sigma(-s_k h_k)$$

$$\Pr\{s_k \to -s_k\} = \frac{1}{1 + \exp(2 s_k h_k / T)}$$

$$= \frac{1}{1 + \exp(\Delta E / T)}$$

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Stability

- Are stochastic Hopfield nets stable?
- Thermal noise prevents absolute stability
- But with symmetric weights:

  average values $\langle s_i \rangle$ become time-invariant

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Does "Thermal Noise" Improve Memory Performance?

- Experiments by Bar-Yam (pp. 316-20):
  - $n = 100$
  - $p = 8$
- Random initial state
- To allow convergence, after 20 cycles set $T = 0$
- How often does it converge to an imprinted pattern?

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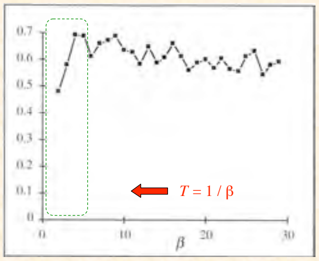
Probability of Random State Converging on Imprinted State (n=100, p=8)



(fig. from Bar-Yam)

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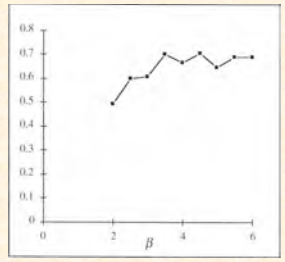
Probability of Random State Converging on Imprinted State (n=100, p=8)



(fig. from Bar-Yam)

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Analysis of Stochastic Hopfield Network

- Complete analysis by Daniel J. Amit & colleagues in mid-80s
- See D. J. Amit, *Modeling Brain Function: The World of Attractor Neural Networks*, Cambridge Univ. Press, 1989.
- The analysis is beyond the scope of this course

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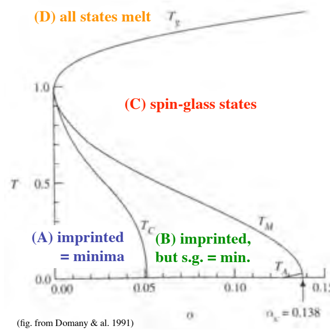
Phase Diagram



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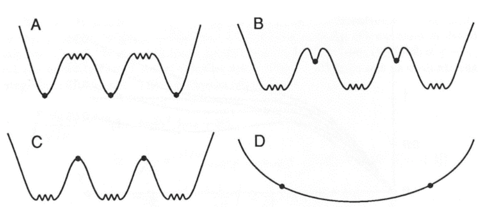
Conceptual Diagrams of Energy Landscape



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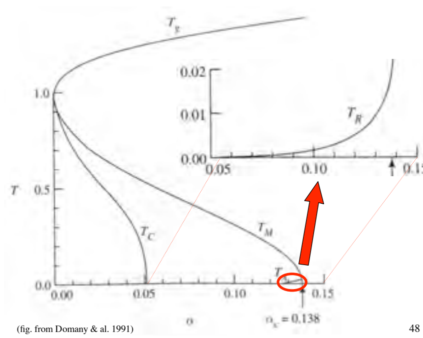
Phase Diagram Detail



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