CS311, Fall 2003
Homework 2 SOLUTIONS. Due Tuesday January 27

Note: I have corrected some minor mistakes.

**Question 1** Expand \((x - 4y)^4\) using the binomial theorem.

**ANSWER:**

\[
(x - 4y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^i (-4y)^{4-i}
\]

\[
= \left( \binom{4}{0} x^0 (-4y)^4 + \binom{4}{1} x^1 (-4y)^3 + \binom{4}{2} x^2 (-4y)^2 + \binom{4}{3} x^3 (-4y)^1 + \binom{4}{4} x^4 (-4y)^0 \right)
= 1 \cdot 1 \cdot 256 \cdot y^4 - 4 \cdot x \cdot 64 \cdot y^3 + 6 \cdot x^2 \cdot 16 \cdot y^2 - 4 \cdot x^3 \cdot 4 \cdot y + 1 \cdot x^4 \cdot 1
= 256y^4 - 256xy^3 + 96x^2y^2 - 16x^3y + x^4
\]

**Question 2** What is the coefficient of \(x^4y^3\) in \((x - 2y)^7\)? Show your work.

**ANSWER:** By the binomial theorem, the term containing \(x^4y^3\) in \((x - 2y)^7\) is \(\binom{7}{4} x^4(-2y)^3\). This gives: \(-\left(\frac{7}{4!}\right) \cdot 8x^4y^3\). So the coefficient is \(-280\).

**Question 3** Determine \(n\) such that \(\binom{n}{5} \binom{n-5}{3} = \frac{n!}{35!}\).

Show your work.

**ANSWER:**

\[
\binom{n}{5} \binom{n-5}{3} = \frac{n!}{n-5)! (n-8)!} \cdot \frac{(n-5)!}{3! (n-8)! (n-9)!}
= \frac{n!}{35! (n-8)!}
\]

This is equal to \(\frac{n!}{35! (n-8)!}\) when \((n - 8)! = 1\)
This happens when \(n - 8 = 0\) or \(n - 8 = 1\). This means \(n = 8\) or \(n = 9\) will work.

Checking, let us plug in 8 for \(n\). \(\binom{8}{5} \binom{8-5}{3} = \binom{8}{3} = \binom{8}{5} = 8!/5!3!\).

**Question 4** Questions concerning combinations with repetition:

(a) How many ways can 15 identical candy bars be distributed among 5 children?

**ANSWER:** This problem assumes that the children may get more than one candy bar. This is a problem of combinations with repetition. The formula gives \(\binom{15+5-1}{15} = \binom{19}{15} = 19 \cdot 18 \cdot 17 \cdot 16 / 5! = 3876\)

(I do not expect you to do this by hand.)

(b) How many ways can 15 identical candy bars be distributed among 5 children if the youngest gets exactly 1?

**ANSWER:** Give one child one candy bar and you are left with 14 candy bars to give to 4 children. The answer to this is \(\binom{14+4-1}{14} = \binom{17}{14} = 17 \cdot 16 \cdot 15 / 3! = 680\).

(c) How many ways can 15 identical candy bars be distributed among 5 children
if the youngest gets exactly 1 or exactly 2?

ANSWER: There are two possibilities. If the youngest gets exactly 1, we have 680 ways.
If the youngest gets exactly two, then that leaves 13 candy bars to give to 4 kids.
There are \( \binom{13+4-1}{13} = \binom{16}{13} = 16 \times 15 \times 14/3! = 560 \) ways to do this.
Thus there are 680 + 560 = 1240 ways to give either 1 or 2 to the youngest.
(d) How many ways can we distribute 4 different candy bars among 4 children if each child gets exactly one?

ANSWER: Since the candy bars are different and each child gets just one, this is just a permutation problem. The answer is 4! = 24.
(e) We have 15 different candy bars. How many ways can we give exactly 1 candy bar to each of 4 children?

ANSWER: Again, this is a permutation of 15 things taken 4 at a time, which is

\[ P(15, 4) = 15!/11! = 15 \times 14 \times 13 \times 12 = 32760. \]

**Question 5** (a) Express \( \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \)
as a summation. Assume \( n \) is an integer and \( n \geq 2 \).

ANSWER: \( \sum_{i=2}^{n} 1/i! \)

(b) Express: \( 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \)
as a summation. Here the summation goes to infinity.

ANSWER: \( \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i!} \)

**Question 6** Evaluate:

(a) \( \sum_{i=1}^{6}(i^2 + 1) \)

ANSWER: \( (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) + (6^2 + 1) \)

\[ = 2 + 5 + 10 + 17 + 26 + 37 = 97 \]

(b) \( \sum_{i=1}^{6} i(-1)^i \)

ANSWER: \( 1(-1)^1 + 2(-1)^2 + 3(-1)^3 + 4(-1)^4 + 5(-1)^5 + 6(-1)^6 \)

\[ = -1 + 2 - 3 + 4 - 5 + 6 = 3 \]

(c) \( \sum_{i=0}^{4} 2^{-i} \) Do this out and also use the formula.

ANSWER: \( 2^0 + 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} = 1 + 1/2 + 1/4 + 1/8 + 1/16 = 31/16 \)

Plugging in 1/2 for \( a \) in \( \sum_{i=0}^{n} a^i = \frac{a^{n+1} - a}{a-1} \)
gives \( 2 - 1/2^n \).

\( = (1 - (1/2)^{n+1})/(1/2) = 2(1 - (1/2)^{n+1}) = 2 - 1/2^n. \)

Plugging in \( n = 4 \) gives: \( 2 - \frac{1}{2^4} = 2 - 1/16 = 32/16 - 1/16 = 31/16. \)

**Question 7** Give an algebraic expression for:

(a) \( \sum_{i=1}^{n}(i + 1) \)

ANSWER: Since \( \sum_{i=1}^{n} i = n(n + 1)/2 \), we have:

\( \sum_{i=1}^{n}(i + 1) = \sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) - 1 = (n + 1)(n + 2)/2 - 1. \)

Which is \( (n^2 + 3n + 2)/2 - 2/2 = (n^2 + 3n)/2. \)

Another way to do this is: \( \sum_{i=1}^{n}(i + 1) = (\sum_{i=1}^{n} i) + \sum_{i=1}^{n} 1) = \)
\[ n(n + 1)/2 + n = (n^2 + n)/2 + 2n/2 = (n^2 + 3n)/2. \]

(b) \[ \sum_{i=2}^{n} 2^i \]

ANSWER: We know \[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1. \] So we need to subtract the terms for \( i = 0 \) and \( i = 1 \), giving \[ 2^{n+1} - 1 - 2^0 - 2^1 = 2^{n+1} - 1 - 1 - 2 = 2^{n+1} - 4. \]