Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
Overview

What is spectral graph theory?

- The study of eigenvalues and eigenvectors of graphs.
- The multiset of eigenvalues of a matrix is called its spectrum.
- If the matrix corresponds to a graph $G$, then the spectrum of the matrix is also called the spectrum of $G$.
- A graph's spectrum provides insight into its structure and properties that it possesses.
Overview
An analogy

- Astronomers discover a new star, and want to learn about its composition.
Overview
An analogy

- Astronomers discover a new star, and want to learn about its composition.
- It’s obviously too far away to gather physical samples, and far too dangerous even if they could get there.
Overview
An analogy

- Astronomers discover a new star, and want to learn about its composition.
- It’s obviously too far away to gather physical samples, and far too dangerous even if they could get there.
- Instead, they can observe the light emitted by the star, its \textit{spectrum}.
Overview
An analogy

- Astronomers discover a new star, and want to learn about its composition.
- It’s obviously too far away to gather physical samples, and far too dangerous even if they could get there.
- Instead, they can observe the light emitted by the star, its *spectrum*.
- They have models that tell them what the spectrum of a star should look like depending on its temperature, and more models that relate temperature to composition.
Overview
An analogy

- Astronomers discover a new star, and want to learn about its composition.
- It’s obviously too far away to gather physical samples, and far too dangerous even if they could get there.
- Instead, they can observe the light emitted by the star, its *spectrum*.
- They have models that tell them what the spectrum of a star should look like depending on its temperature, and more models that relate temperature to composition.

\[
\text{stars } \leftrightarrow \text{ graphs } \\
\text{prisms, telescopes, models } \leftrightarrow \text{ linear algebra, graph theory}
\]
Let $A \in \mathbb{R}^{n \times n}$ be a real, square matrix with $n$ rows and $n$ columns.

The matrix $A$ represents a linear transformation between $n$-dimensional vectors.

In other words, given $x \in \mathbb{R}^n$, there exists a $y \in \mathbb{R}^n$ such that

$$Ax = y$$

If $y = \lambda x$ for some scalar $\lambda$, then $x$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$. We can rewrite the above equation as

$$Ax = \lambda x$$
Linear Algebra Primer

So what?

▶ So... eigenvalues and eigenvectors just let us rewrite an equation?

▶ Yes! But this rewriting shows exactly what makes eigenvectors interesting.

▶ Linear transformations typically both rotate and scale vectors when they are applied.

▶ Eigenvectors are special in that they are only scaled, and their eigenvalue is the scaling factor.
Linear Algebra Primer

Visual Example

Consider applying a linear transformation to every point of a 2D grid.

The red vector (and the pixel it points to) gets rotated and stretched. But the blue vector is an eigenvector, since its direction remains unchanged.
Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
History
Eigenvalues and Eigenvectors: 1700s

- Originally arose in the study of physical systems.
- Leonhard Euler noted the importance of principal axes in the rotation of rigid bodies.
- Joseph-Louis Lagrange later discovered that these principle axes are the eigenvectors of the inertia matrix.
History

Eigenvalues and Eigenvectors: 1800s

- Augustin-Louis Cauchy extended and generalized the work of Euler and Lagrange. Referred to eigenvalues as characteristic roots.

- Joseph Fourier extended the existing work to differential equations, allowing him to solve the heat equation in 1822.

- Eigenvalues and eigenvectors of matrices start to be studied for their own sake.

- Cauchy proves that real symmetric matrices have real eigenvalues.
Eigenvalues and Eigenvectors: 1900s

- David Hilbert first used the German word *eigen*, meaning “own”, “inherent”, or “characteristic”, in reference to eigenvalues in 1904, which eventually replaced the then popular term of proper values.

- Hilbert extended the concept to infinite matrices.

- First numerical algorithms for finding eigenvalues developed by Richard von Mises in 1929 (power method).
History
Spectral Graph Theory

- Appeared as a branch of algebraic graph theory in the 1950s and 1960s.

- Early work focused on using the adjacency matrix, which limited initial results to regular graphs.

- Research was independently begun in quantum chemistry, as eigenvalues of graphical representation of atoms correspond to energy levels of electrons.

- In 1973, Czech mathematician Miroslav Fiedler developed the concept of algebraic connectivity (2nd eigenvector of Laplacian). The corresponding eigenvector is named after him.
Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
Some More Linear Algebra

- In general, eigenvalues and eigenvectors can have imaginary components.

- Real, symmetric $n \times n$ matrices will always have $n$ real eigenvalues, and a corresponding set of $n$ linearly independent eigenvectors.

- Eigenvectors are not unique. A scalar multiple of an eigenvector is also an eigenvector.

- The spectrum of a matrix is generally a multiset. Eigenvalues can be repeated.
Matrix Representation of Graphs

Assume $d_i$ is the degree of vertex $i$. There are three matrix representations are commonly used:

- The adjacency matrix

  $$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

- The Laplacian matrix

  $$L_{ij} = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

- The normalized Laplacian matrix

  $$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } d_i \neq 0 \\ -\frac{1}{\sqrt{d_i d_j}} & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$
Adjacency Matrix

- Adjacency matrices were the first matrices to be explored in spectral graph theory.
- They work nicely with regular graphs, but results proved difficult to generalize to general graphs.
- We’ll denote the eigenvalues of the adjacency matrix $A$ corresponding to graph $G$ as

\[ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \]
Laplacian Matrix

- The eigenvalues of the Laplacian and adjacency matrices behave similarly as long as the graph is regular.

- Once the Laplacian became popular, spectral graph theory became much more widely applicable.

- The vector of all ones is always an eigenvector of $L$, with corresponding eigenvalue of 0.

- We’ll denote the eigenvalues of the Laplacian matrix $L$ corresponding to graph $G$ as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$
Normalized Laplacian Matrix

- A degree-normalized version of the Laplacian.
- Closely related to the *walk* or *diffusion* matrix of a graph, another matrix representation of a graph that reflects the dynamics of a random walk on.
- We won’t be discussing this matrix in detail.
What can we learn from a graph’s spectrum?

Basic Facts

- $\alpha_1$ is at least as large as the average degree, and no larger than the maximum degree.
- If $G$ is connected, then $\alpha_1 > \alpha_2$.
- The vector of all ones is an eigenvector of $A$ with eigenvalue $\alpha_1$ iff $G$ is an $\alpha_1$–regular graph.
- The number of times 0 appears in the spectrum of $L$ is equal to the number of connected components of $G$.
- $\lambda_n$ is at most twice the maximum degree of a vertex.
- $\alpha_n = -\alpha_1$ iff $G$ is bipartite.
What can we learn from a graph’s spectrum?
How to draw the graph.

▶ We can use the second and third eigenvectors to as coordinates for vertices, which can often lead to nice visual representations of graphs.
▶ Doesn’t always work out!
Algebraic Connectivity

- $\lambda_2$ grows as the graph is more connected.
- We can use the corresponding eigenvector to partition the graph.
- Suppose $v_2$ is the eigenvector of $L$ corresponding to $\lambda_2$, the Fiedler vector.
- Fiedler proved that if we select a value $t \leq 0$, then the set of vertices $i$ such that $v_2(i) \geq t$ form a connected component.
Isospectral Graphs

- It’s possible for two non-isomorphic graphs to have the same spectrum.
- The graphs may look very different, but share fundamental properties.
- A graph’s spectrum is invariant to relabeling of vertices, so of course isomorphic graphs are also isospectral.
Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
Applications

- Gaining insight into diffusion processes on a graph.
- Identifying important vertices (PageRank), bottlenecks, and other graph structures.
- Spectral clustering of data using a similarity matrix of the data.
- Putting bounds on graph properties that can be efficiently computed.
- Many applications in physics and other sciences in general.
Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
Open Problems

▶ **Conjecture:** If $A$ is the adjacency matrix of a $d$-regular graph, then there is a symmetric signing of $A$ (replacing $+1$ entries with $-1$) so that the resulting matrix has all eigenvalues of magnitude at most $2\sqrt{d - 1}$.

▶ Are almost all graphs determined by their spectrum?
Outline

Overview

Linear Algebra Primer

History

Theory

Applications

Open Problems

Homework Problems

References
Homework Problems

1. Find the (Laplacian) spectrum of the Peterson graph.
2. Find the (Laplacian) spectrum of $K_n$.
3. Prove that a graph’s spectrum is invariant under relabeling the vertices.
References

- http://www.cs.yale.edu/homes/spielman/561/
- http://www.math.ucsd.edu/~fan/research/revised.html