Measure and Integration

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Cardinality

Let S be a set
Example: S = { 0, 1, 2 }
Cardinality $|S|$ of discrete set S is number of its elements
Example: $|\{0, 1, 2\}| = 3$

Power set

Let S be a set
Example: S = { 0, 1, 2 }
The power set P(S) is the set of all subsets of S
Example: P(S) = { {}, {0}, {1}, {2}, {0,1}, {1,2}, {2,0}, {0,1,2} }
Cardinality $|P(S)|$ of power set is $2^{|S|}$
Example: $|P(\{0,1,2\})| = 2^{|\{0,1,2\}|} = 2^3 = 8$

Power set

The set $\mathbb{Z}$ of integers is infinite, but countable
Every element of $\mathbb{Z}$ can be given an integer subscript
The set $\mathbb{R}$ of real numbers is not countable
Cardinality of reals is greater than cardinalty of integers
$|\mathbb{R}| > |\mathbb{Z}|$
The “infinite” cardinality $\aleph_0$ (“aleph-null”) represents $|\mathbb{Z}|$
The “infinite” cardinality $\aleph_1$ (“aleph-one”) represents $|\mathbb{R}|$
Power sets: $|P(\mathbb{Z})| = 2^{\aleph_0} = \aleph_1$

Cardinality

What is the cardinality of a segment?
How many points are in a segment?
Infinitely many: $\aleph_1$
So two segments have the same cardinality
How can you compare two segments?
Two rectangles? Two triangles? Two boxes? Two tetrahdra?
Cardinality is inadequate
Use length or area or volume or ..

Length

Consider the set S_1 of line segments in the plane
Each segment $s_1 \in S_1$ is an element of $P(\mathbb{R}^2)$
Each segment $s_1 \in S_1$ has a finite length $L(s_1)$
Arrow notation: L maps subsets to the non-negative reals
$L : P(\mathbb{R}^2) \rightarrow \mathbb{R}^+$

Area

Consider the set S_2 of rectangles in the plane
Each rectangle $s_2 \in S_2$ is an element of $\mathrm{P}(\mathbb{R}^2)$
Each rectangle $s_2 \in S_2$ has a finite area $A(s_2)$
$A : P(\mathbb{R}^2) \rightarrow \mathbb{R}^+$

Area

Example: $s_2 = [a, a+1] \times [b, b+1]$
$A(s_2) = 1 \times 1 = 1$

Area

Example: $s_2 = [a, a + w] \times [b, b + h]$
$A(s_2) = w \times h = wh$

Area

Example: $s_2 = [x, x + dx] \times [y, y + dy]$
$A(s_2) = dx\ dy$
Better: $dA(s_2) = dx\ dy$

Area

Example: $p = (x, y)$ (a single point)
$A(p) = 0$
Area of a point is zero

Area

Example: s_1 (subset of a line)
$A(s_1) = 0$
Area of a segment is zero

Volume

Consider the set S_3 of boxes (3-cells) in the plane
Each 3-cell is an element of $\mathrm{P}(\mathbb{R}^2)$
Each 3-cell $s_3 \in S_3$ has a finite volume $V(s_3)$
$V : P(\mathbb{R}^2) \rightarrow \mathbb{R}^+$
Note: $V(s_3) = 0$ within $\mathbb{R}^2$

Measure

Measure μ generalizes idea of length, area, volume
$\mu(S) = 0$ means S has measure zero
Measure of empty set is zero
$\mu(\{\}) = 0$
Measure is non-negative
$\mu : P(S) \to \mathbb{R}^+$
Measure of union is sum minus intersection
$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

Measure

Let S be the union of s[i]: $S = s[1] \cup s[2] \cup .. \cup s[n]$

$$S = \bigcup_{i=1}^{n} s[i]$$

If sets s[1], s[2], .. s[n] are disjoint, their intersection is empty

$s[i] \cap s[j] = \{\}$ for $i \neq j$

Measure of S is the sum of the measures of its disjoint s[i]

$$\mu(S) = \mu(s[1]) + \mu(s[2]) + .. + \mu(s[n])$$

$$\mu(S) = \sum_{i=1}^{n} \mu(s[i])$$

Continuous set

Consider S, the union of x-values from *a* to *b*

$S = [a, b]$

Measure $\mu([a, x])$ = x-a gives length of segment in S

If subsets are not countable, S can't be represented as s[i]

S can be represented by countable union of disjoint intervals

$S = [a, a + dx) \cup [a + dx, a + 2dx) \cup .. \cup [b - dx, b]$

Diferential measure $d\mu = dx$

Integration

Measure of S is the "sum" of the measures of its elements

$$\mu(S) = d\mu([a, a+dx) + .. + d\mu([b-dx, b])$$

$$\mu(S) = \int_{x \in S} d\mu(x) = \int_{x=a}^{b} d\mu(x)$$

Integration

Measure can be weighted by some real-valued w(x)

$w : S \rightarrow \mathbb{R}^+$

Example: $w(x) = x^2$

$$\int_{x \in [0,1]} x^2 \, d\mu([0,1]) = \int_{x=0}^{1} x^2 dx$$

Define a new measure ν such that $d\nu = x^2 \, dx$

Integral can be written $\int_{x=0}^{1} d\nu$

In this new measure, the length is small near $x = 0$

Average

The average of a function depends on the measure

$$\text{average(f)} = \frac{\text{weighted measure}}{\text{measure}} = \frac{\int_{x \in S} f(x)\ d\mu}{\int_{x \in S} d\mu}$$

Example: average of $f(x) = 1$ over interval [a,b]

$$\text{avg} = \frac{\int_{x \in [a,b]} 1\ d\mu}{\int_{x \in [a,b]} d\mu} = 1$$

Average

$$\text{average(f)} = \frac{\int\limits_{x \in S} f(x)\ d\mu}{\int\limits_{x \in S} d\mu}$$

Example: average of $f(x) = x$ over interval [a,b]

$$\text{avg} = \frac{\int\limits_{x \in [a,b]} x\ d\mu}{\int\limits_{x \in [a,b]} d\mu} = \frac{\int\limits_{x=a}^{b} x\ dx}{\int\limits_{x=a}^{b} dx} = \frac{(\frac{x^2}{2})|_{x=a}^{b}}{x|_{x=a}^{b}} = \frac{\frac{b^2}{2} - \frac{a^2}{2}}{b-a} = \frac{\frac{(b+a)(b-a)}{2}}{b-a} = \frac{(a+b)}{2}$$

Area

Set S = {plane} = { (x,y) }
What measure μ on the space S?
“Obvious” choice: dμ = dx dy

Lines

Set S = {lines in the plane}
Equation y = mx + b has parameters m and b
Set S = { (m, b) }
What measure on the space S?
“Obvious” choice: dμ = dm db
But consider all the lines from mx+b to (m+dm)x+b
Lines form a bundle with angle
$\phi(m) = \tan^{-1}(m + dm) - \tan^{-1}(m)$
In general, $\phi(m_1) \neq \phi(m_2)$

Lines

What measure on the space S of lines in plane?
Second choice: dμ = dϕ db
But consider all the lines with angle ϕ, y-intercept b
Bundle of lines has different thickness db depending on ϕ
How thick? Compare $db(\phi)$ to $db(0)$
$db(\phi) = |\cos(\phi)|\, db(0)$

Lines

What measure on the space S of lines in plane?
Third choice: $d\mu \ \cos\phi \, d\phi \, db$
Invariant under rotation and translation
Other measures can be derived from this one
Example: $d\mu = \frac{dm \ db}{(1+m^2)^{3/2}}$