

Repeated Roots: Differentiation

Method 2: Differentiation

ex $F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+10} + \frac{k_3}{(s+10)^2}$

k_1 & $k_3 \rightarrow$ normal cover-up method

$$k_1 = \left. \frac{32s(s+1)}{(s+10)^2} \right|_{s=-2} = 1$$

$$k_3 = \left. \frac{32s(s+1)}{(s+2)} \right|_{s=-10} = -360$$

for k_2 : multiply both sides by $(s+10)^2$, then differentiate w.r.t. s before plugging in $s = -10$

$$\left[\frac{d}{ds} \left[\frac{32s(s+1)}{s+2} \right] \right] \Big|_{s=-10} = \left[\frac{d}{ds} \left[\frac{k_1}{s+2} (s+10)^2 + \frac{k_2}{s+10} (s+10)^2 + k_3 \right] \right] \Big|_{s=-10}$$

$$= \left[\cancel{g(s)^2} + \cancel{g'(s)}(s+10)^2 + k_2 + 0 \right] \Big|_{s=-10}$$

$$k_2 = \left[\frac{d}{ds} \left[\frac{32s^2 + 32s}{s+2} \right] \right] \Big|_{s=-10} = \frac{(64s+32)(s+2) - (32s^2+32s)}{(s+2)^2} \Big|_{s=-10} = 31$$

$$F(s) = \frac{1}{s+2} + \frac{31}{s+10} + \frac{-360}{(s+10)^2}$$

$$F(s) = \frac{1}{(s+1)^3} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k_3}{(s+1)^3} = \frac{As^2 + Bs + C}{(s+1)^3}$$

Differentiation \rightarrow find k_2 w/ multiply by $(s+1)^3$,
 d/ds , plug in $s = -1$

find k_1 by multiply by $(s+1)^2$,

$\frac{d^2}{ds^2}$, plug in $s = -1$

PFE: Complex Roots

ex
 $F(s) = \frac{1}{s^2 - 2s + 2}$

roots @ $\frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2} = 1 \pm j$

Note: for real circuits & signals, complex roots (and their residues) will always show up in conjugate pairs

$$F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - (1 + j))(s - (1 - j))} = \frac{k_1}{s - (1 + j)} + \frac{k_1^*}{s - (1 - j)}$$

Option 1: Do nothing different

$$k_1 = \left. \frac{1}{s - (1 - j)} \right|_{s=1+j} = \frac{1}{2j} = \frac{-j}{2}$$

(could skip knowing $k_2 = k_1^*$)
 $k_2 = \left. \frac{1}{s - (1 + j)} \right|_{s=1-j} = \frac{1}{-2j} = \frac{j}{2} = k_1^*$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{j}{2} \mathcal{L}^{-1}\left\{ \frac{-1}{s - (1 + j)} + \frac{1}{s - (1 - j)} \right\} =$$

$$\begin{aligned} f(t) &= \frac{j}{4} (e^{(1+j)t} u(t) + e^{(1-j)t} u(t)) \\ &= \frac{j}{2} e^t (e^{jt} + e^{-jt}) u(t) = e^t \left(\frac{e^{jt} - e^{-jt}}{2j} \right) u(t) \\ &= e^t \sin(t) u(t) \end{aligned}$$

Option 2: let's do the general case?

$$\begin{aligned} &= \mathcal{L}^{-1}\left\{ \frac{k}{s - (\sigma + j\omega)} + \frac{k^*}{s - (\sigma - j\omega)} \right\} \\ &= k e^{(\sigma + j\omega)t} u(t) + k^* e^{(\sigma - j\omega)t} u(t) \\ &= e^{\sigma t} (k e^{j\omega t} + k^* e^{-j\omega t}) u(t) \\ &= e^{\sigma t} (\operatorname{Re}\{k\} (e^{j\omega t} + e^{-j\omega t}) + j \operatorname{Im}\{k\} (e^{j\omega t} - e^{-j\omega t})) u(t) \\ &= e^{\sigma t} (\operatorname{Re}\{k\} 2 \cos(\omega t) - \operatorname{Im}\{k\} 2 \sin(\omega t)) u(t) \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin \theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \end{aligned}$$

$$= 2e^{\sigma t} \left(\sqrt{\operatorname{Re}\{k\}^2 + \operatorname{Im}\{k\}^2} \cos(\omega t - \tan^{-1} \left(\frac{\operatorname{Im}\{k\}}{\operatorname{Re}\{k\}} \right)) \right) u(t)$$

$$= 2e^{\sigma t} (|k| \cos(\omega t - \angle k)) u(t)$$

$$\boxed{\frac{k}{s - (\sigma + j\omega)} + \frac{k^*}{s - (\sigma - j\omega)} = 2e^{\sigma t} |k| \cos(\omega t - \angle k) u(t)}$$

Option 3: Manipulate polynomial into one of the existing items in the table