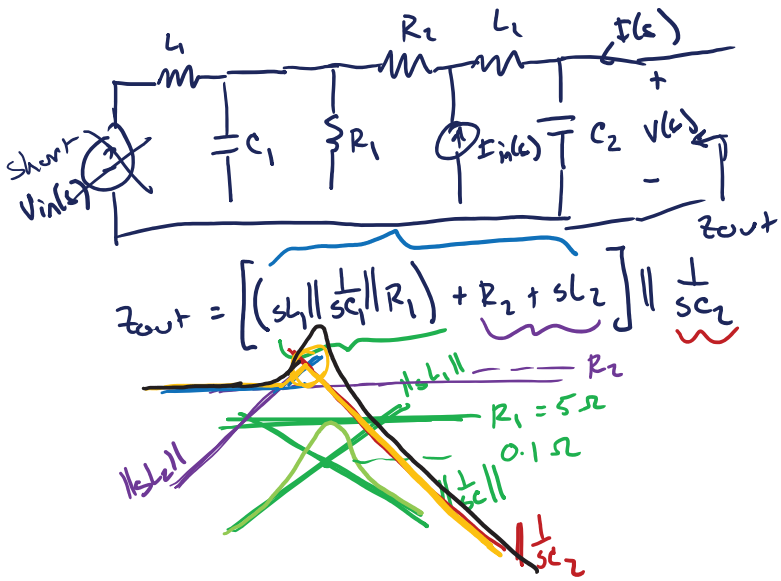


$$Z_{out} = \left[\left(sL_1 \| \frac{1}{sC_1} \| R_1 \right) + R_2 + sL_2 \right] \| \frac{1}{sC_2}$$

L_1 100mH
C_1 1mF
R_1 5Ω
L_2 16mH
C_2 10mF
R_2 50Ω

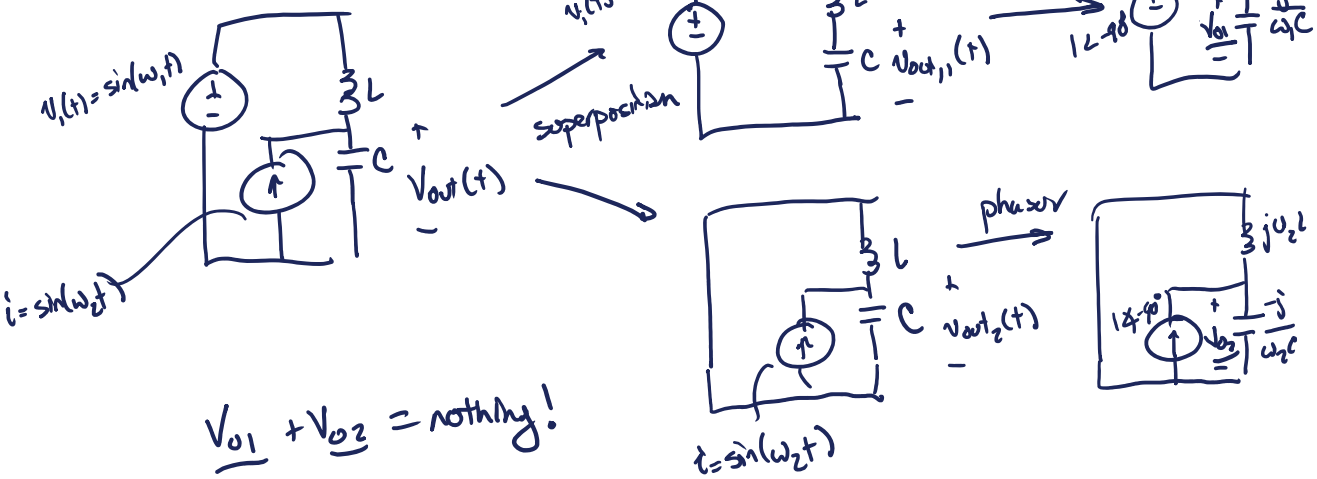
Transfer function

$$|H(s)| = \frac{y(s)}{u(s)}$$

all other indep inputs set to zero

$$Z_{out} = \frac{V(s)}{I(s)}$$

Two sources @ different freq in phasor analysis?



$V_{o1} + V_{o2} =$ nothing!

$$v_{out} = |V_{o1}| \cos(\omega_1 t + \angle V_{o1}) + |V_{o2}| \cos(\omega_2 t + \angle V_{o2})$$

Bode plots w/ constants

$$H(s) = \frac{4}{s^3 + 7s^2 + 12s} = \frac{4}{s(s^2 + 7s + 12)}$$

$$= \frac{1/3}{s\left(\frac{s^2}{12} + \frac{7}{12}s + 1\right)}$$

$$\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{Q\omega_0} + 1$$

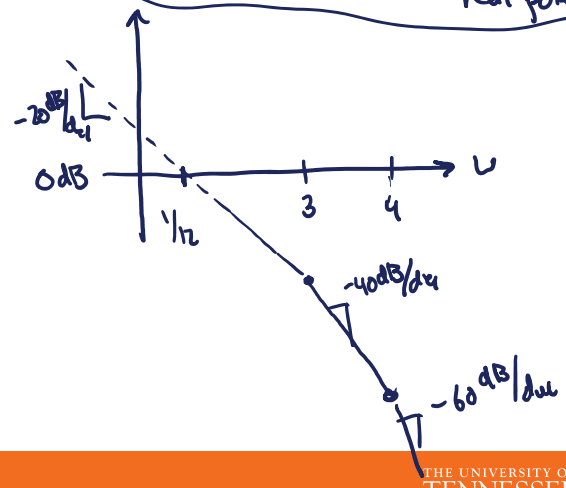
$$\frac{1}{12} = \frac{1}{\omega_0^2} \qquad \omega_0 = \sqrt{12}$$

$$\frac{7}{12} = \frac{1}{Q\omega_0} \qquad Q = \frac{1}{\sqrt{12}}\frac{12}{7} = \frac{\sqrt{12}}{7} < 0.5$$

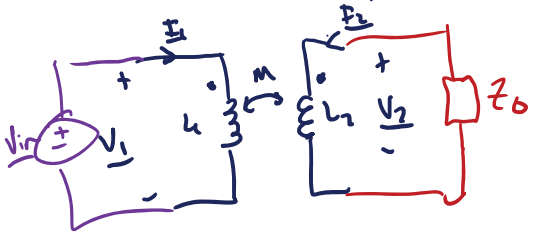
two distinct real poles

$$\frac{4}{s(s+3)(s+4)} = \frac{\frac{4}{3\cdot 4}}{s\left(\frac{s}{3}+1\right)\left(\frac{s}{4}+1\right)}$$

$$\left|H(j\omega)\Big|_{\omega \to \phi}\right| \approx \left|\frac{1}{12} \cdot \frac{1}{j\omega}\right| = \frac{1}{12\omega}$$



phasor domain



$$\begin{cases} \underline{V}_1 = j\omega L_1 \underline{I}_1 + j\omega M \underline{I}_2 \\ \underline{V}_2 = j\omega L_2 \underline{I}_2 + j\omega M \underline{I}_1 \end{cases}$$

$$\underline{V}_2 = -\underline{I}_2 Z_0 \longleftrightarrow \underline{I}_2 = \frac{-\underline{V}_2}{Z_0}$$

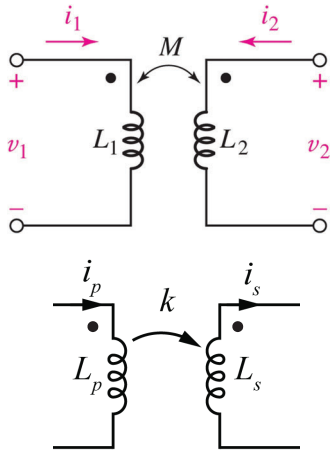
$$\underline{V}_1 = \underline{V}_{in}$$

Course Content

- Magnetically Coupled Circuits (Ch 13)
- Sinusoidal Steady-State Analysis (Ch 10)
- AC Circuit Power Analysis (Ch 11)
- Fourier Circuit Analysis (Ch 17)
- Circuit AC Analysis in the s-Domain (Ch 14)
- Frequency Response (Ch 15)
- Polyphase Circuits (Ch 12)
- Two-Port Networks (Ch 16)

Ch 13 – Magnetically Coupled Circuits

Coupled Inductors



Defining Equations

$$v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2(t) = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

Dot convention

- Current into the dot on one terminal produces a positive open circuit voltage w.r.t. the dot on the other

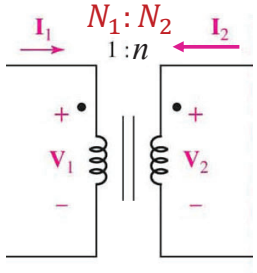
Coupling Coefficient

$$k = \frac{M}{\sqrt{L_p L_s}}$$

Recall

- Equivalent circuits

Ideal Transformer



Defining Equations

$$\frac{V_1}{N_1} = \frac{V_2}{N_2} = \cdots$$

$$0 = N_1 I_1 + N_2 I_2 + \cdots$$

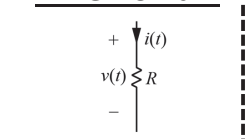
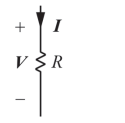
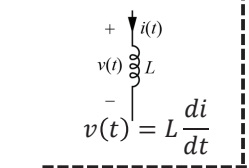
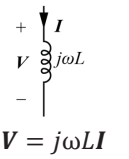
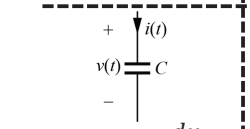
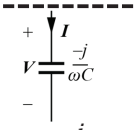
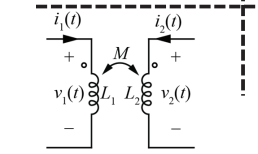
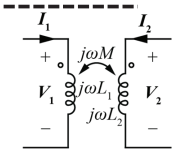
Coupled inductors with:

- no energy storage (L→ ∞)
- Perfect coupling (*k=1*)

Recall

- Z/V/I reflection

Ch 10 – Sinusoidal Steady State

Time Domain	Phasor Domain
	
$v(t) = i(t)R$	$\boldsymbol{V} = \boldsymbol{I}R$
	
$v(t) = L\frac{di}{dt}$	$\boldsymbol{V} = j\omega L\boldsymbol{I}$
	
$i(t) = C\frac{dv}{dt}$	$\boldsymbol{V} = \frac{-j}{\omega C}\boldsymbol{I}$
	
$v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$ $v_2(t) = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$	$\boldsymbol{V}_1 = j\omega L_1\boldsymbol{I}_1 + j\omega M\boldsymbol{I}_2$ $\boldsymbol{V}_2 = j\omega M\boldsymbol{I}_1 + j\omega L_2\boldsymbol{I}_2$

Phasor Notation

$$A\cos(\omega t + \varphi) \Leftrightarrow A\angle\varphi$$
$$= Re\{Ae^{j(\omega t+\varphi)}\}$$
$$= Re\{A\cos(\omega t + \varphi) + jA\sin(\omega t + \varphi)\}$$

Impedance and Admittance

$$Z = R + jX$$

Impedance — Resistance — Reactance

$$Y = \frac{1}{Z} = G + jB$$

Admittance — Conductance — Susceptance

Circuit Analysis

- Real circuits always have all real signals in the time domain
- All 201 analysis techniques apply
- Gives only forced/steady-state/particular response, for single sinusoidal source
- Phasor superposition

Ch 11 – AC Power Analysis

Average (DC) Power: $P = \int_{-\infty}^{\infty} p(t)dt$ For periodic signals: $P = \int_{t_0}^{t_0+T} p(t)dt$

Average power in a resistor: $P_R = \left[\sqrt{\int i(t)^2 dt}\right]^2 R$ For sinusoids: $I_{rms} = \frac{I_A}{\sqrt{2}}$

$I_{rms} = I_{eff}$

Sinusoidal Power

$$p(t) = [V_A\cos(\omega t + \varphi_V)][I_A\cos(\omega t + \varphi_I)]$$

$$= \underbrace{\frac{V_A I_A}{2}\cos(2\omega t + \varphi_V + \varphi_I)}_{\text{Double-frequency}} + \underbrace{\frac{V_A I_A}{2}\cos(\varphi_V - \varphi_I)}_{\text{DC = average}}$$

$$\frac{V_A I_A}{2} = V_{rms}I_{rms}$$

Complex Power

$$S = \frac{\boldsymbol{V}\boldsymbol{I}^*}{2} = \boldsymbol{V}_{rms}\boldsymbol{I}_{rms}$$

$$S = P + jQ$$

Complex [VA] — Real/Avg [W] — Reactive/Quad. [VAR]

Apparent Power: $|S| = \frac{V_A I_A}{2} = V_{rms}I_{rms}$ Power Factor: $PF = \frac{P}{|S|}$ (leading/lagging)

Impedance match for max power transfer: $Z_L = Z_{th}{}^*$

Ch 17 – Fourier Series and Transform

https://en.wikipedia.org/wiki/Fourier_transform

Fourier Series:

For periodic $f(t)$ with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin(k\omega_0 t)\, dt$$

Fourier Transform:

For periodic or non-periodic $f(t)$

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} F(\omega) d\omega$$

- Frequency content of a signal
- Gives mag/phase of sinusoids that add up to original signal

Ch 14 – Laplace Transform

Unilateral Laplace Transform: $F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt$

Inverse Laplace Transform: $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{st} F(s) ds$, σ_0 in ROC (to the right of all poles in complex plane)

Complex frequency = Laplace variable = $s = \sigma + j\omega$

Laplace transform is a **linear transformation**. Other properties and transforms in tables

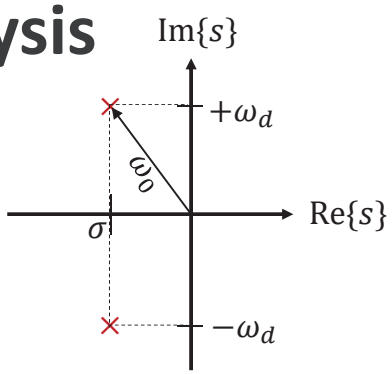
Inverse Transforms: *Long Division* → *Factor* → *PFE* → *Tables*

PFE Special cases:

Repeated: $\frac{N(s)}{(s+5)^2} = \frac{k_1}{s+5} + \frac{k_2}{(s+5)^2}$ (Differentiation or coefficient matching)

Complex: $\frac{N(s)}{s^2+4} = \frac{k_1 s + k_2}{s^2+4} = \frac{k_1}{s-j2} + \frac{k_1^*}{s+j2}$

Ch 14 – Laplace Circuit Analysis



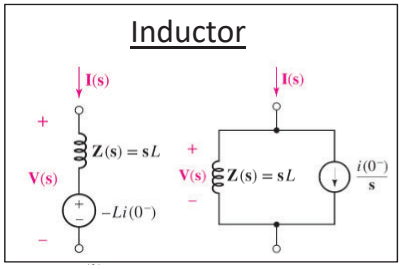
Poles and Zeros:

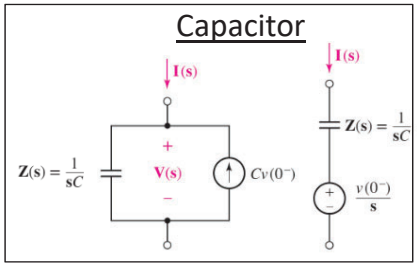
$$F(s) = \frac{N(s)}{D(s)}$$

Zeros: roots of $N(s)$

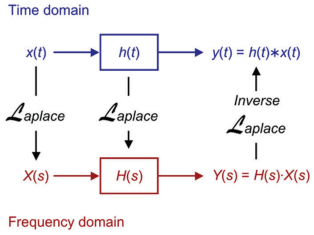
Poles: roots of $D(s)$

Circuit Transformation:





Convolution:



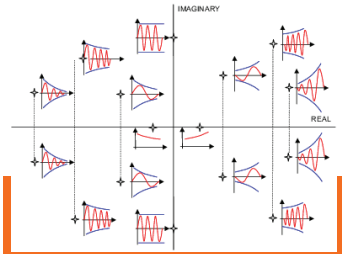
$$v_o(t) = v_i(t) * h(t) = \int_{-\infty}^{\infty} v_i(t-\tau)h(\tau)d\tau$$

Transfer Functions:

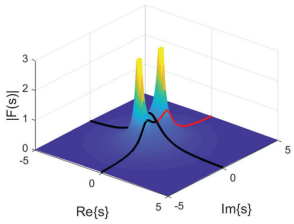
$$H(s) = \frac{V_o(s)}{V_i(s)}$$

Poles of $H(s)$ define "form" of terms in natural response of the circuit

Poles of $V_i(s)$ define "form" of terms in forced response of the circuit



Ch 15 – Frequency Response



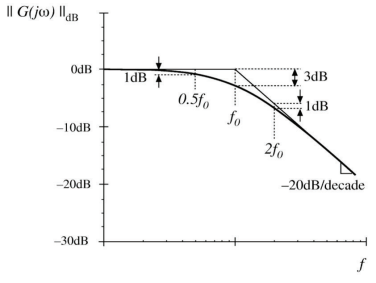
Frequency Response: $H(s \to j\omega)$

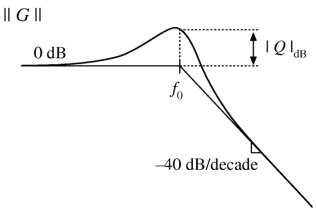
- Circuit response to sinusoidal inputs
- Valid if all poles in LHP

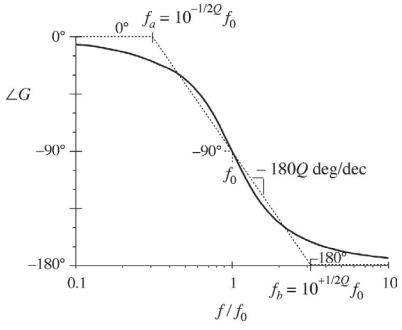
Bode plot: mag-phase plots on log-log axes

- $\|H(j\omega)\|_{dB} = 20\log(|H(j\omega)|)$
- $\measuredangle H(j\omega)$

Templates and Approximations:







Filter Design

- Bandwidth, graphical analysis, Chebyshev and Butterworth, Sallen-Key Amplifier
- Resonant circuits

FUTURE TOPICS

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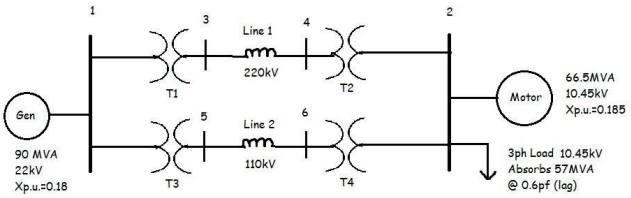
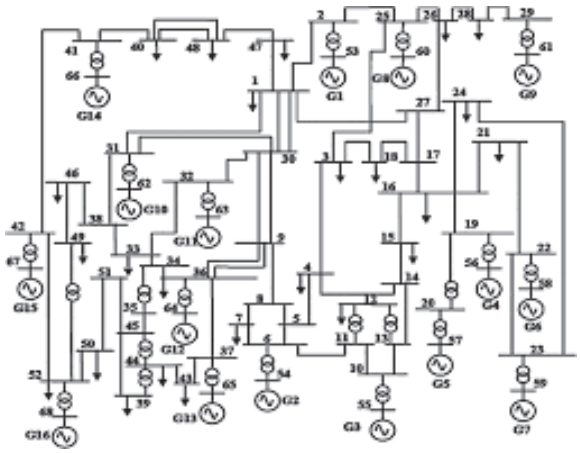
Power Systems

- ECE 325 -- Electric Energy System Components





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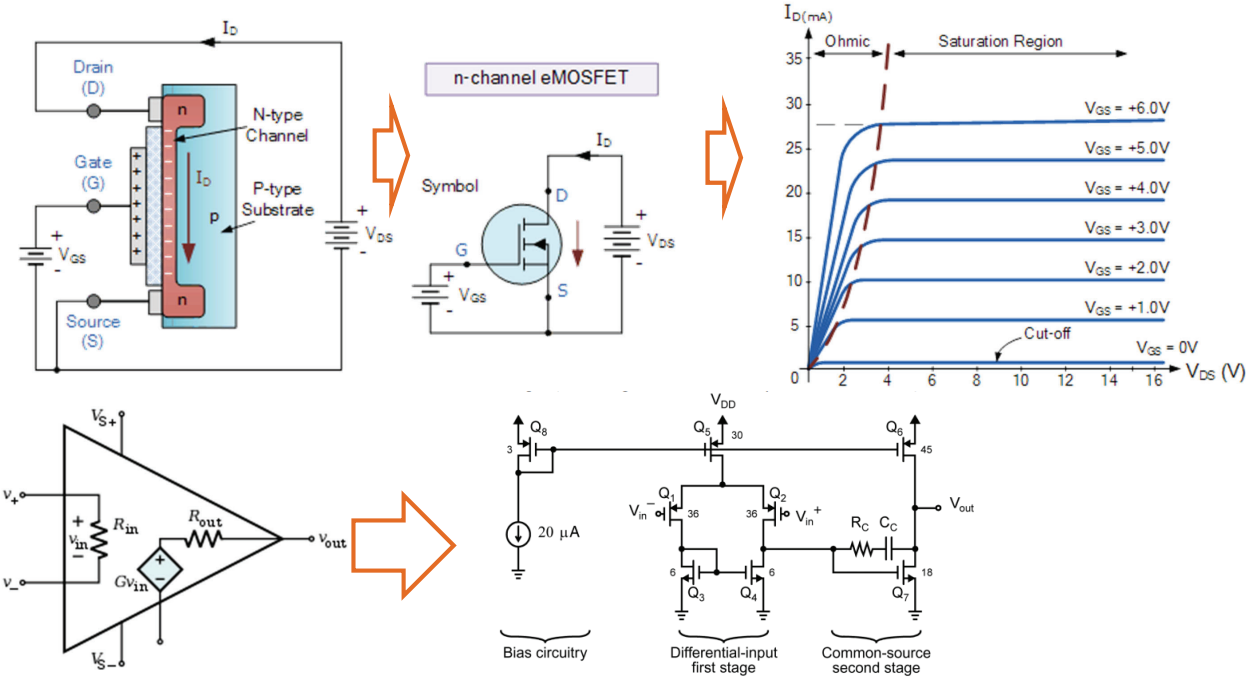
Nonlinear Circuits

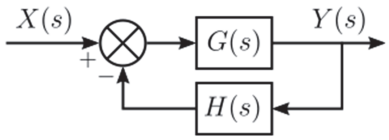
- ECE 335 & 336 – Electronic Devices



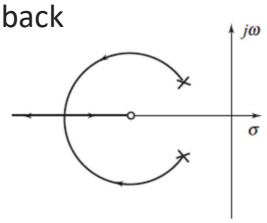
Closed-Loop Control

- ECE 316 – Signals and Systems

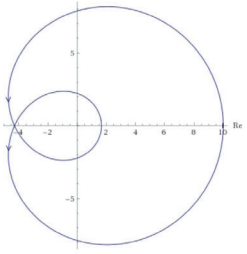
Classic Control



Root Locus: How poles and zeros move as you change feedback



Nyquist Plots: Frequency response in the complex plane



Modern Control

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

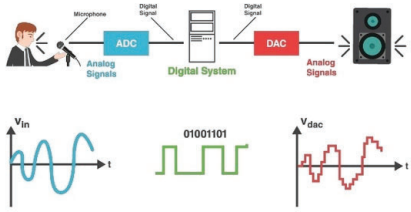
$$H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

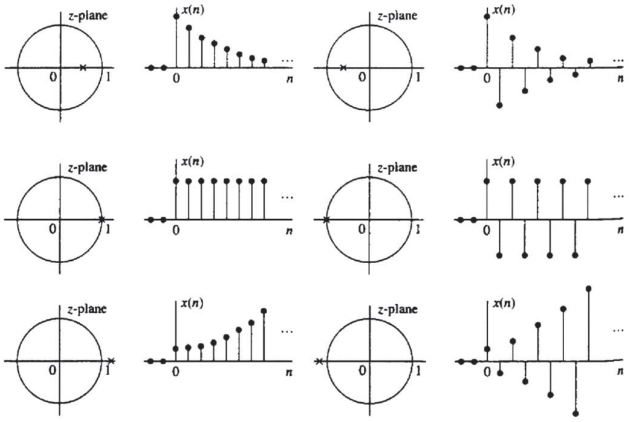
Digital / Discrete Time Signals

- ECE 315 – Signals and Systems

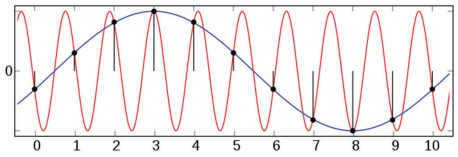
Cont. time $\sum_n A_n \frac{d^n}{dt^n} v_o(t) = \sum_m B_m \frac{d^m}{dt^m} v_i(t) \xrightarrow{\mathcal{L}} \sum_n A_n s^n V_o(s) = \sum_m B_m s^m V_i(s)$

Discrete time $\sum_n A_n y[k-n] = \sum_m B_m u[k-m] \xrightarrow{z} \sum_n A_n z^{-n} Y(z) = \sum_m B_m z^{-m} U(z)$





Sampling and aliasing:



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PART I: COUPLED INDUCTORS AND PHASORS

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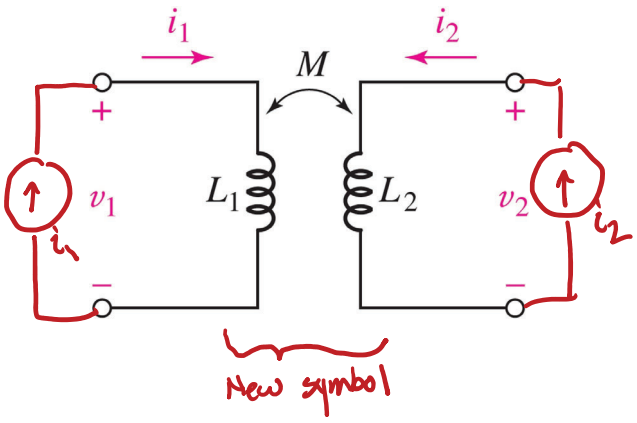
Chapter 13, Lectures 2-5

MUTUAL INDUCTANCE

Mutual Inductance



Apply superposition:

when $i_2 = 0$

$$v_1 = L_1 \frac{di_1}{dt}$$

$$v_2 = M \frac{di_1}{dt}$$

when $i_1 = 0$

$$v_1 = M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt}$$

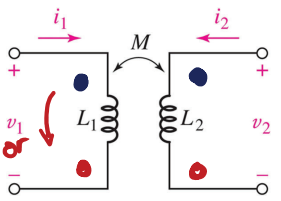
$$v_1 = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}$$

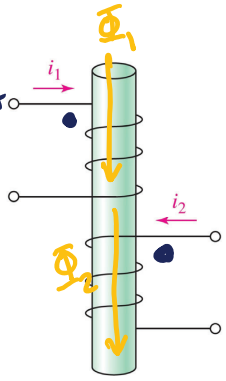
Sign depends on winding arrangement

Symbols and Dot Convention

Note: Always assign passive sign convention

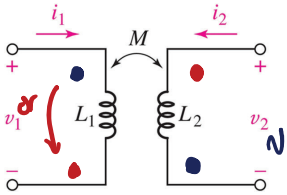


$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$

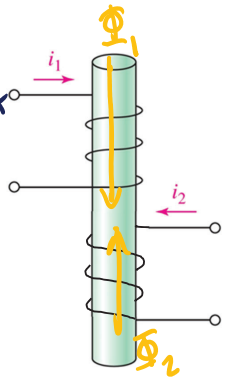


Physically:

If both winding currents enter the dotted terminals, the fluxes are additive



$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$



Circuit:

Current flowing into the dotted terminal of one winding will produce an open-circuit voltage that is positive at the dotted terminal of the other

Coupling Coefficient

Define $k = \frac{M}{\sqrt{L_1 L_2}}$ ≡ "coupling coefficient"

Always $0 \le |k| \le 1$

$k = 0$ → two seperate inductors

$\vdots$ } coupled inductors

$k = 1$ → perfect coupling → "transformer"

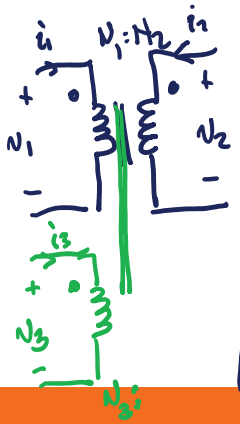
Ideal Transformer

L_1 & L_2 are "large" $\quad k=1$

Recall: Inductors & Transformers cannot have DC voltage applied to any winding
- $v = L\frac{di}{dt}$ → const. voltage implies constant, non-zero $\frac{di}{dt}$ & $i \to \pm\infty$
- $v = N\frac{d\Phi}{dt}$ → only get voltage from time-varying signals
- Materials: any core material saturates at some maximum flux/Area

$E = \frac{1}{2}L_{oo}I^2 = \frac{1}{2}L_{oo}\left(\frac{1}{L_{oo}}\int_0^t v\,dt\right)^2 \to \frac{1}{2}\frac{1}{L_{oo}}\int_0^t v\,dt \to E_{stored} \approx \phi$

Ideal transformer w/ L_1 & L_2 "large" has no energy storage



$$i_1 v_1 + i_2 v_2 = \phi$$

$$i_1 \cancel{v_2}\frac{N_1}{\cancel{N_2}} + i_2 \cancel{v_2} = \phi$$

$$N_3 i_3 + N_1 i_1 + N_2 i_2 = \phi$$

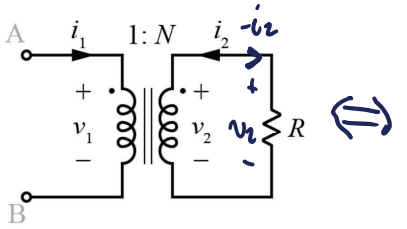
$$\frac{v_3}{N_3} = \frac{v_1}{N_1} = \frac{v_2}{N_2}$$

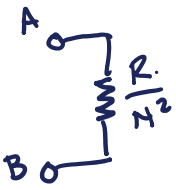
>2 windings

sum of scaled currents equal

scaled voltages are equal

$v_1 = \frac{v_2}{N}$ & $i_1 + N i_2 = \phi \to i_1 = -N i_2$

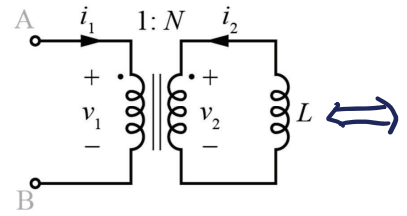


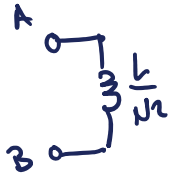


$$v_2 = -i_2 R$$

$$N v_1 = -\left(-\frac{i_1}{N}\right)R$$

$$\boxed{v_1 = i_1 \frac{R}{N^2}}$$

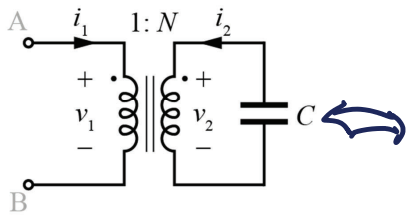


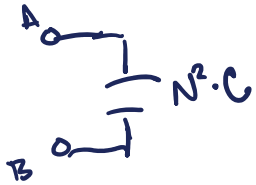


$$v_2 = L\frac{d}{dt}(-i_2)$$

$$N v_1 = L\frac{d}{dt}\left(\frac{i_1}{N}\right)$$

$$\boxed{v_1 = \frac{L}{N^2}\frac{di_1}{dt}}$$



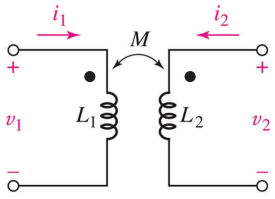


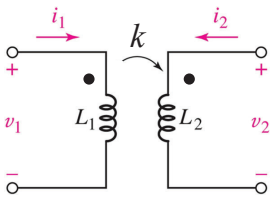
$$-i_2 = C\frac{dv_2}{dt}$$

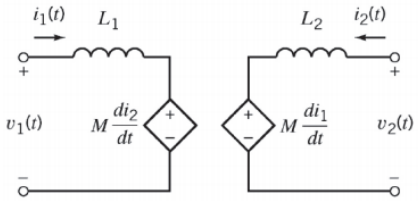
$$\frac{i_1}{N} = C\frac{d}{dt}N v_1$$

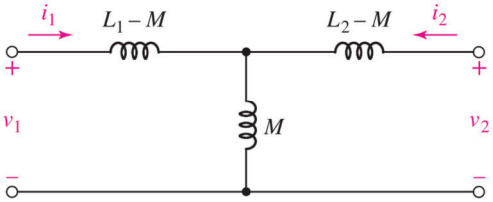
$$\boxed{i_1 = N^2 C \frac{dv_1}{dt}}$$

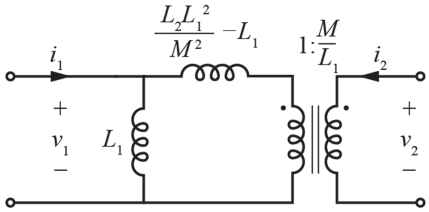
Equivalent Circuits

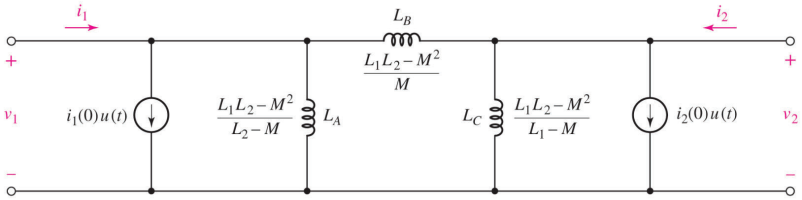












Chapter 10, Lectures 6-11

PHASOR TRANSFORM

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Complex Numbers (Review)

$j = \sqrt{-1}$, $j^2 = -1$

$z = x + jy = re^{j\theta}$

rectangular form — polar form

$$\begin{cases} |z| = r = \sqrt{x^2 + y^2} \equiv \text{magnitude} \\ \angle z = \theta = \tan^{-1}\left(\frac{y}{x}\right) \equiv \text{phase} \end{cases}$$

$$\begin{cases} Re\{z\} = x = r\cos\theta \\ Im\{z\} = y = r\sin\theta \end{cases}$$

Euler's Formula:

$$re^{j\theta} = r\cos\theta + jr\sin\theta$$

Complex Conjugate: $z^* = x - jy$ (same real, swap sign on imag)

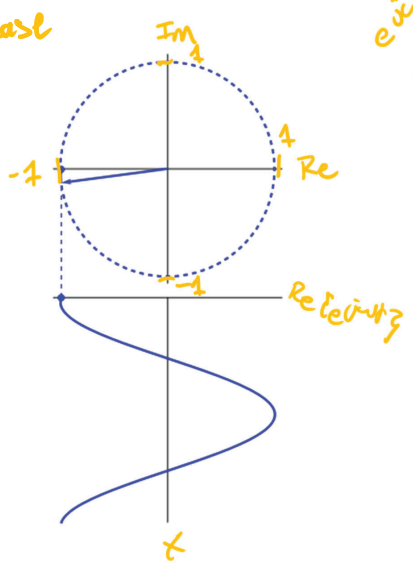
Sinusoids as Complex Numbers

Euler: $re^{j\theta} = r\cos\theta + jr\sin\theta$

$Re\{re^{j\theta}\} = r\cos\theta$

$$v(t) = A\cos(\omega t + \varphi)$$
$$= Re\{A e^{j(\omega t + \varphi)}\}$$
$$= Re\{A e^{j\omega t} e^{j\varphi}\}$$

- $Re\{\}$: project onto Real axis (it's a cosine)
- A: Amplitude
- $e^{j\omega t}$: sinusoid @ freq ω
- $e^{j\varphi}$: phase



$e^{j\omega t}$ → magnitude = 1

phase varies with time

$\theta = \omega t$

θ goes 0 to 2π as t goes 0 to $\frac{2\pi}{\omega}$

Phasor Transformation

Phasor: A complex number that represents a sinusoid

useful in circuits with **single-frequency sinusoidal** inputs

$A\cos(\omega t + \phi)$

- Amplitude A
- ~~frequency~~ ω → known for single-frequency problems
- phase ϕ
- ~~function~~ cos → by convention always use cos
- ~~time~~ t → steady-state solutions

$v(t) = A\cos(\omega t + \phi) = \text{Re}\{A e^{j\omega t} e^{j\phi}\}$

Phasor transformation ↓

$\underline{V} = A e^{j\phi} \iff A \angle \phi$ (short hand)

"bar" is bold in both

ex $v_2(t) = B\sin(\omega t + \theta)$

$v_2(t) = B\cos(\omega t + \theta - 90°)$

transform ↓

$\underline{V}_2 = B e^{j(\theta - \frac{\pi}{2})} \iff B \angle \theta - \frac{\pi}{2}$

Phasor Circuit Elements

Time Domain	Phasor Domain
$v(t) = i(t)R$	$\underline{V} = \underline{I}R$
$v(t) = L\frac{di}{dt}$	$\underline{V} = j\omega L\underline{I}$
$i(t) = C\frac{dv}{dt}$	$\underline{I} = j\omega C\underline{V}$

phasor transform

$v(t) = A\cos(\omega t + \phi) \longrightarrow \underline{V} = A e^{j\phi}$

$i(t) = \frac{A}{R}\cos(\omega t + \phi) \longrightarrow \underline{I} = \frac{A}{R} e^{j\phi}$

$v(t) = i(t)R$

$\boxed{\underline{V} = \underline{I}R}$

$i(t) = A\cos(\omega t + \phi) \longrightarrow \underline{I} = A e^{j\phi}$

$v(t) = L A\omega \cos(\omega t + \phi + 90°) \longrightarrow \underline{V} = L A\omega e^{j(\phi + \frac{\pi}{2})}$

$v(t) = L\frac{di}{dt}$

$\underline{V} = j\omega L A e^{j\phi}$

$\boxed{\underline{V} = j\omega L\underline{I}}$

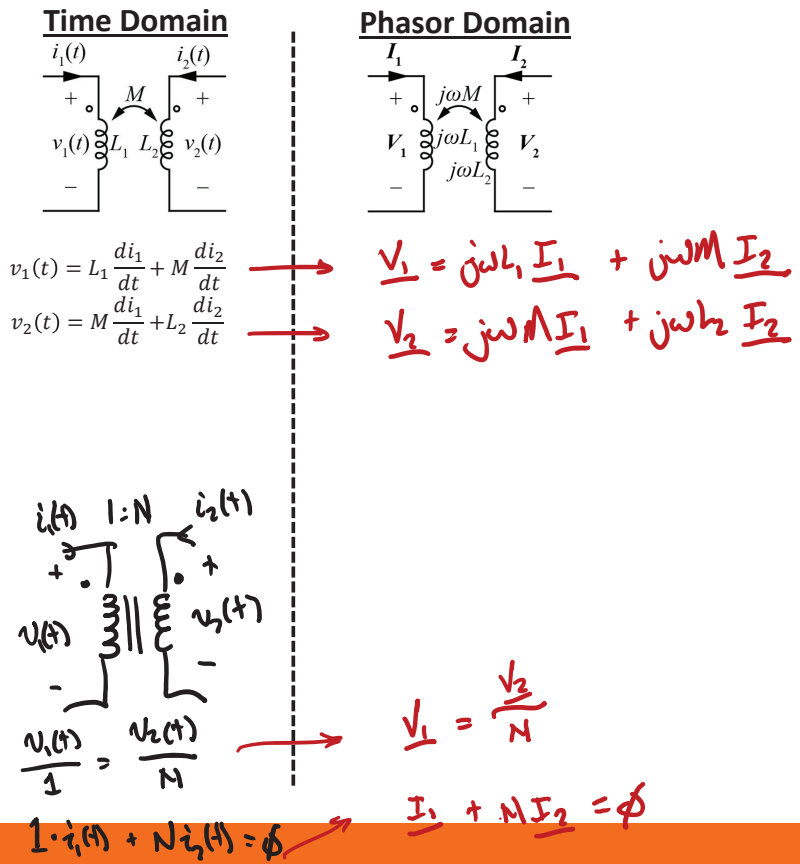
$v(t) = A\cos(\omega t + \phi) \longrightarrow \underline{V} = A e^{j\phi}$

$i(t) = C A\omega \cos(\omega t + \phi + 90°) \longrightarrow \underline{I} = C A\omega e^{j(\phi + \frac{\pi}{2})}$

$i(t) = C\frac{dv}{dt}$

$\underline{I} = j\omega C A e^{j\phi}$

$\boxed{\underline{V} = \frac{-j}{\omega C}\underline{I}} = \boxed{\underline{V} = \frac{1}{j\omega C}\underline{I}}$



Impedance

Phasor equivalent of ohm's law

$$V = I Z$$

Z = "Impedance" = $R + jX$

$Re\{Z\} = R$ "Resistance"

$Im\{Z\} = X$ "Reactance"

$$\begin{cases} Z_R = R, & \text{Resistor} \\ Z_L = j\omega L, & \text{Inductor} \\ Z_C = \frac{-j}{\omega C}, & \text{Capacitor} \end{cases}$$

All have units of ohms (Z, R, X)

Y = "Admittance" = $\frac{1}{Z} = G + jB$ → All units of siemens

G: conductance

B: susceptance

$Y = \frac{1}{Z} = \frac{1}{R + jX} \neq \frac{1}{R} + j\frac{1}{X}$ No.

Yes. ↳ $\frac{1}{R+jX} \frac{(R-jX)}{(R-jX)} = \frac{R-jX}{R^2+X^2} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2}$

Phasor Circuit Analysis

Pre-condition: circuit with single-frequency sinusoidal inputs & want to solve steady-state solution

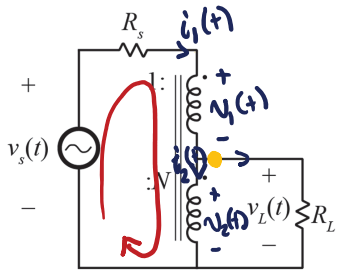
(1) transform all sources & signals into phasor equivalents

(2) transform all passives (L, R, C, M) into impedances Z

(3) Solve circuit using any/all ECE 201 techniques
series/parallel equivalents, source transform, KVL & KCL, Node voltage, mesh current, voltage divider

(4) Transform solution back into time domain (if needed)

Example Problem



$\omega = 2\pi 60$

Find $v_L(t)$ for $v_s(t) = 170\cos(2\pi 60t)$ and for $R_s = 10\ \Omega$, $N = 0.1$, and $R_L = 50\ \Omega$

- Ideal XF & two resistors → no DEs → phasor analysis not very helpful (but do it anyway)

(1) $v_s(t) = 170\cos(\omega t) \longrightarrow \underline{V}_s = 170\angle 0^\circ = 170e^{j0}$

$v_L(t) = V_L\cos(\omega t + \varphi_L) \longrightarrow \underline{V}_L = V_L\angle\varphi_L$

(2) $R_s \rightarrow Z_{R_s} = R_s$ $\quad R_L \rightarrow Z_{R_L} = R_L$

(3) Transformer equations: $\underline{V}_1 = \frac{\underline{V}_2}{N}$ $\quad \underline{I}_1 = -N\underline{I}_2$

Loop $\underline{V}_s = \underline{V}_1 + \underline{V}_2 + \underline{I}_1 R_s$

$\underline{V}_s = \underline{V}_L\left(\frac{1}{N} + 1\right) + \underline{I}_1 R_s$

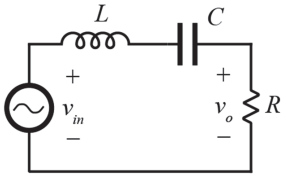
Node $\underline{I}_1 = \underline{I}_2 + \frac{\underline{V}_L}{R_L}$

$\underline{I}_1 = -\frac{1}{N}\underline{I}_1 + \frac{\underline{V}_L}{R_L}$

$\underline{I}_1 = \frac{\underline{V}_L}{R_L}\frac{1}{1+\frac{1}{N}}$

$\underline{V}_s = \underline{V}_L\left(\frac{1}{N} + 1 + \frac{R_s}{R_L}\frac{1}{1+\frac{1}{N}}\right)$

Resonance Example



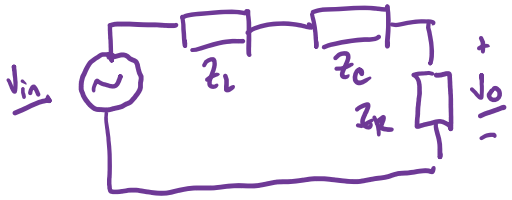
Find $v_o(t)$ for $v_{in}(t) = 10\sin(\omega t)$ and $\omega = 2\pi 100$ kHz, $R = 10\ \Omega$, $L = 10\ \mu$H, and $C = 253$ nF

(1) $v_{in}(t) \rightarrow V_{in} = 10 \angle 90°$

$v_o(t) \rightarrow V_o = A \angle \gamma$

(2) $R \rightarrow Z_R = R = 10\Omega$ $\quad L \rightarrow Z_L = j\omega L = j2\pi$ $\quad C \rightarrow Z_C = \frac{-j}{\omega C} = -j6.28$

$j(2\pi 100e3)(10e-6)$



$$V_o = \frac{Z_R}{Z_R + Z_L + Z_C} V_{in}$$

by resistor voltage divider formula.

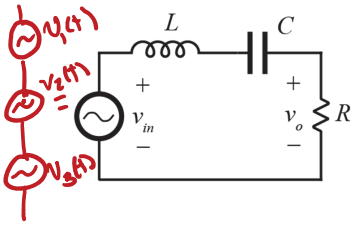
special case $Z_L \simeq -Z_C = j6.28 \rightarrow Z_L + Z_C = \phi$ in this case

$$V_o = V_{in}\frac{Z_R}{Z_R} \rightarrow V_o = V_{in}$$

$V_o = V_{in} = 10 \angle -90° \rightarrow v_o(t) = 10\sin(\omega t)$

Phasor Superposition

Not a single-frequency problem!



Find $v_o(t)$ for $v_{in}(t) = \underbrace{10\sin(\omega t)}_{v_1(t)} + \underbrace{10\sin(2\omega t) + 10\sin(0.5\omega t)}_{v_2(t) \quad v_3(t)}$ and $\omega = 2\pi 100$ kHz, $R = 10\ \Omega$, $L = 10\ \mu$H, and $C = 253$ nF

Apply superposition! must be done in time domain

$v_{in}(t) = v_1(t) + v_2(t) + v_3(t)$

$v_o(t) = v_{o1}(t) + v_{o2}(t) + v_{o3}(t)$

Note: Impedance values change with frequency

(1) $\omega_1 = 2\pi 100$kHz

(2) $\omega_2 = 2\pi 200$kHz

(3) $\omega_3 = 2\pi 50$kHz

$$V_o = V_{in}\frac{Z_R}{Z_R + Z_C + Z_L}$$

$V_{o1} = 10 \angle 90°$

$v_{o1}(t) = 10\sin(\omega t)$

$= 10\cos(\omega t - 90°)$

$V_{o2} = 7.3 \angle -132°$

$v_{o2}(t) = 7.3\sin(2\omega t - 43°)$

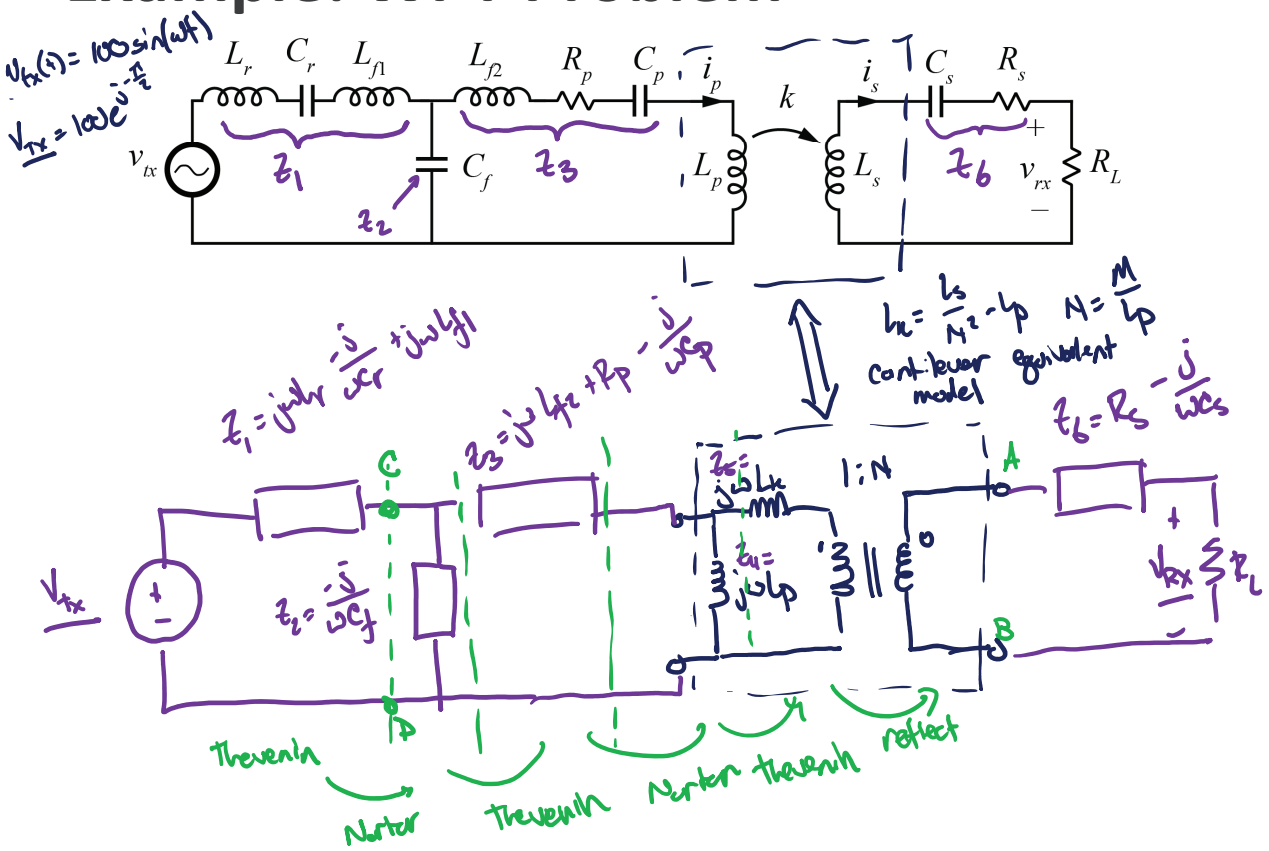
$= 7.3\cos(2\omega t - 132°)$

$V_{o3} = 7.3 \angle -47°$

$v_{o3}(t) = 7.3\sin(\frac{\omega}{2}t + 43°)$

$= 7.3\cos(\frac{\omega}{2}t - 47°)$

Example: WPT Problem



Chapter 10: Lectures 11-14

PHASOR POWER

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RMS of a sinusoid

$i(t) = I_A \cos(\omega t + \phi_I)$ period $T = \frac{1}{f} = \frac{2\pi}{\omega}$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T \left[I_A \cos(\omega t + \phi_I)\right]^2 dt}$$

Trig Identity: $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$I_{rms}^2 = \frac{\omega}{2\pi} I_A^2 \int_0^{2\pi/\omega} \cos^2(\omega t + \phi_I)\, dt$$

$$= \frac{\omega}{2\pi} I_A^2 \int_0^{2\pi/\omega} \frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\phi_I)\, dt$$

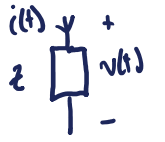
$$= \frac{\omega}{2\pi} I_A^2 \left[\frac{1}{2}t + \frac{1}{2}\sin(2\omega t + 2\phi_I)\frac{1}{2\omega}\right]\Bigg|_0^{2\pi/\omega}$$

$$= \frac{\omega}{2\pi} I_A^2 \left[\frac{1}{2}\left(\frac{2\pi}{\omega} - 0\right) + \frac{1}{4\pi}(\phi)\right]$$

$$I_{rms}^2 = \frac{I_A^2}{2} \rightarrow \boxed{I_{rms} = \frac{I_A}{\sqrt{2}}}$$

for sinusoidal signals: $X_{rms} = \frac{X_A}{\sqrt{2}}$

Power with Sinusoidal Sources



In sinusoidal steady-state with single-frequency ω

$v(t) \rightarrow \underline{V} = V_A e^{j\phi_V}$

$i(t) \rightarrow \underline{I} = I_A e^{j\phi_I}$

$\Big\} \; \underline{V} = \underline{I}\,\underline{Z}$

$V_A e^{j\phi_V} = I_A e^{j\phi_I} |Z| e^{j\phi_Z}$

$\begin{cases} V_A = I_A |Z| \\ \phi_V = \phi_I - \phi_Z \\ \hookrightarrow \phi_Z = \phi_V - \phi_I \end{cases}$

Power: $p(t) = i(t) \cdot v(t)$ is not LTI

$$p(t) = \mathrm{Re}\{I_A e^{j\omega t} e^{j\phi_I}\} \cdot \mathrm{Re}\{V_A e^{j\omega t} e^{j\phi_V}\}$$

Go back to time domain:

$$p(t) = V_A \cos(\omega t + \phi_V) \cdot I_A \cos(\omega t + \phi_I)$$

Trig Identity: $2\cos\theta\cos\varphi = \cos(\theta+\varphi) + \cos(\theta-\varphi)$

$$p(t) = V_A I_A \frac{1}{2}\left[\cos(2\omega t + \phi_V + \phi_I) + \cos(\phi_V - \phi_I)\right]$$

$\cos(2\omega t + \phi_V + \phi_I)$: sinusoid @ 2ω

$\cos(\phi_V - \phi_I)$: constant, not time-varying

Average Power $\boxed{P = \frac{V_A I_A}{2}\cos(\phi_V - \phi_I)}$

Complex Power

$\varphi_z = \varphi_V - \varphi_I$

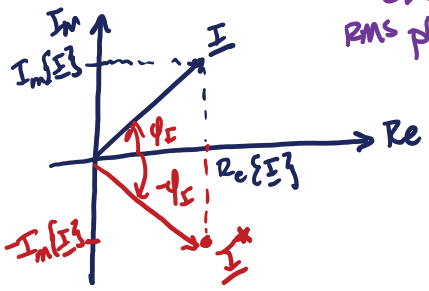
$$P = \frac{V_A I_A}{2}\cos(\varphi_V - \varphi_I) = V_{rms} I_{rms}\cos(\varphi_V - \varphi_I) = \frac{1}{2}\mathrm{Re}\{\underline{V}\underline{I}^*\} = \frac{1}{2}\mathrm{Re}\{\underline{V}_{rms}\underline{I}^*_{rms}\}$$

RMS phasors

$$\frac{1}{2}\mathrm{Re}\{\underline{V}\underline{I}^*\} = \frac{1}{2}\mathrm{Re}\{V_A e^{j\varphi_V} I_A e^{-j\varphi_I}\}$$

$$= \frac{1}{2}\mathrm{Re}\{V_A I_A e^{j(\varphi_V - \varphi_I)}\}$$

$$= \frac{1}{2} V_A I_A \cos(\varphi_V - \varphi_I)$$



what about the imaginary part of $\underline{V}\underline{I}^*$?

$$S = \frac{1}{2}\underline{V}\underline{I}^* = P + jQ$$

- S: Complex power [V·A]
- P: Real or Average Power [W]
- Q: Reactive or Quadrature power [VAR]

Apparent Power & Power Factor

"Apparent Power" $= |S| = \frac{1}{2}V_A I_A = V_{rms} I_{rms}$

Maximum value that real power can be for a given V_A & I_A

"Power factor" $= PF = \frac{P}{|S|} = \frac{P}{\frac{1}{2}V_A I_A} = \frac{P}{V_{rms} I_{rms}} = \cos(\varphi_V - \varphi_I)$

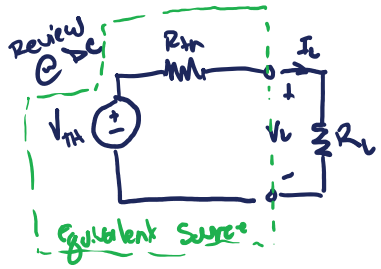
$0 < PF < 1$

"leading" → capacitive

"lagging" → inductive

current w.r.t. voltage

Maximum Power Transfer



what value of R_L will give maximum $P_L = I_L V_L$

$R_L = R_{th}$ for max power.

$$P_L = I_L \cdot V_L = \left(\frac{V_{th}}{R_{th}+R_L}\right)\left(V_{th}\frac{R_L}{R_L+R_{th}}\right) = V_{th}^2\frac{R_L}{(R_L+R_{th})^2}$$

$$\frac{\partial P_L}{\partial R_L} = \phi = V_{th}^2\left[\frac{1}{(R_L+R_{th})^2} + \frac{-2R_L}{(R_L+R_{th})^3}\right]$$

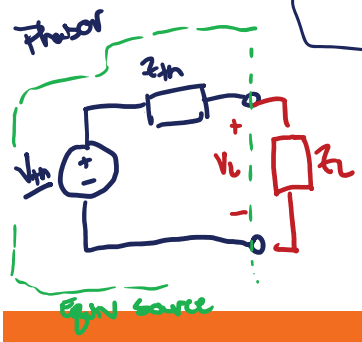
$$\phi = \frac{R_L + R_{th} - 2R_L}{(R_L+R_{th})^3} \qquad 0 = R_{th} - R_L \quad \therefore R_L = R_{th}$$ ✓

If R_L is fixed, what value of R_{th} gives maximum P_L? $R_{th} = \phi$

Maximizing power may not always be good.

$$\text{Efficiency} = \frac{P_{out}}{P_{in}} = \frac{P_L}{P_{th}} = \frac{I_L^2 R_L}{I_L^2 R_L + I_L^2 R_{th}} = \frac{1}{2} = 50\% \text{ when } R_L = R_{th}$$

if $R_L \gg R_{th}$ → high efficiency but lower power

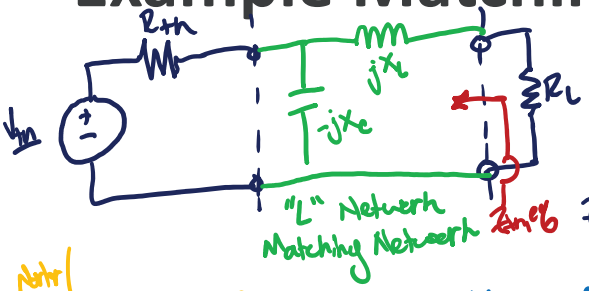


what value of Z_L maximizes power P_L

$$Z_L = Re\{Z_{th}\} - jIm\{Z_{th}\}$$

$$\boxed{Z_L = Z_{th}^*}$$

Example Matching Circuits



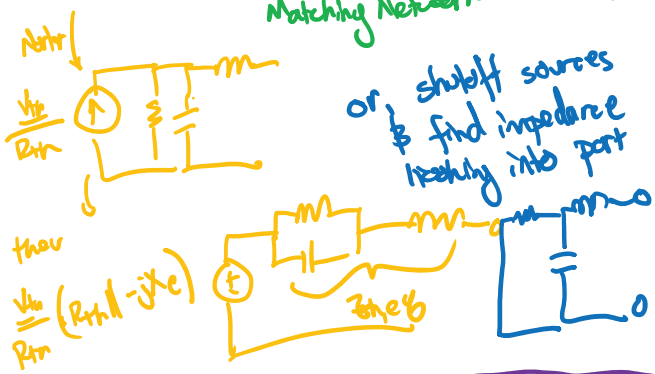
$R_{th} > R_L$

$X_L = \omega L$ $\quad X_c = \frac{1}{\omega C}$

$$Z_{th,eq} = jX_L + (R_{th} \parallel -jX_c)$$

$$= jX_L + \frac{-jX_c R_{th}}{R_{th} - jX_c}\cdot\frac{(R_{th}+jX_c)}{(R_{th}+jX_c)}$$

$$= \left[\frac{X_c^2 R_{th}}{R_{th}^2 + X_c^2}\right] + j\left[\frac{-X_c R_{th}^2}{R_{th}^2 + X_c^2} + X_L\right]$$



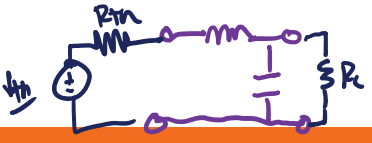
for max power $Z_{th,eq} = R_L^* = R_L$

① set $Re\{Z_{th,eq}\} = \frac{X_c^2 R_{th}}{R_{th}^2 + X_c^2} = R_L$ get X_c

② set $Im\{Z_{th,eq}\} = 0 = \frac{-X_c R_{th}^2}{R_{th}^2 + X_c^2} + X_L$

Circuit shown can work if $R_{th} > R_L$

If we instead had $R_{th} < R_L$ (flip same L-Network)



PART II: FOURIER AND LAPLACE

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Chapter 17, Lectures 20-22

FOURIER SERIES AND TRANSFORM

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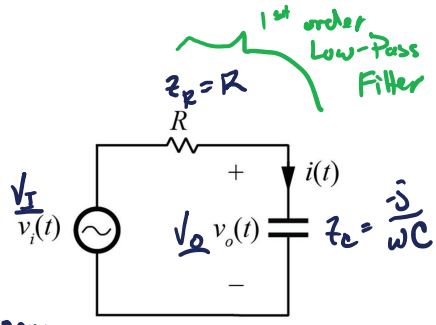
Frequency Response



$$V_o = V_I \frac{Z_c}{Z_R + Z_c} = V_I \frac{\frac{-j}{\omega C}}{R - \frac{j}{\omega C}} = V_I \frac{1}{j\omega CR + 1}$$

$$\frac{V_o}{V_I} = \frac{1}{j\omega RC + 1} \rightarrow \frac{V_o}{V_I} \equiv H(j\omega) \equiv \text{Frequency Response}$$

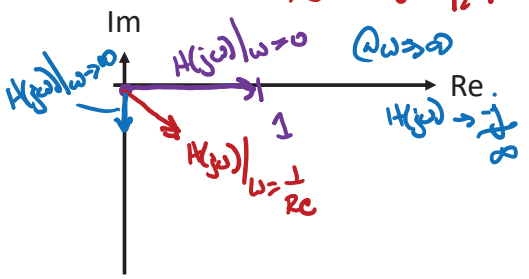
$H(j\omega)$ is the Frequency Response. It is some complex number with ω as a variable that tells us the gain & phase shift of our output compared to the input

$$H(j\omega) = |H| \angle (\angle H)$$

$$H(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} \angle -\tan^{-1}\left(\frac{\omega RC}{1}\right)$$

@ $\omega = 0 \rightarrow H(j\omega) = 1$

@ $\omega = \frac{1}{RC} \rightarrow H(j\omega) = \frac{1}{\sqrt{2}} \angle -45°$



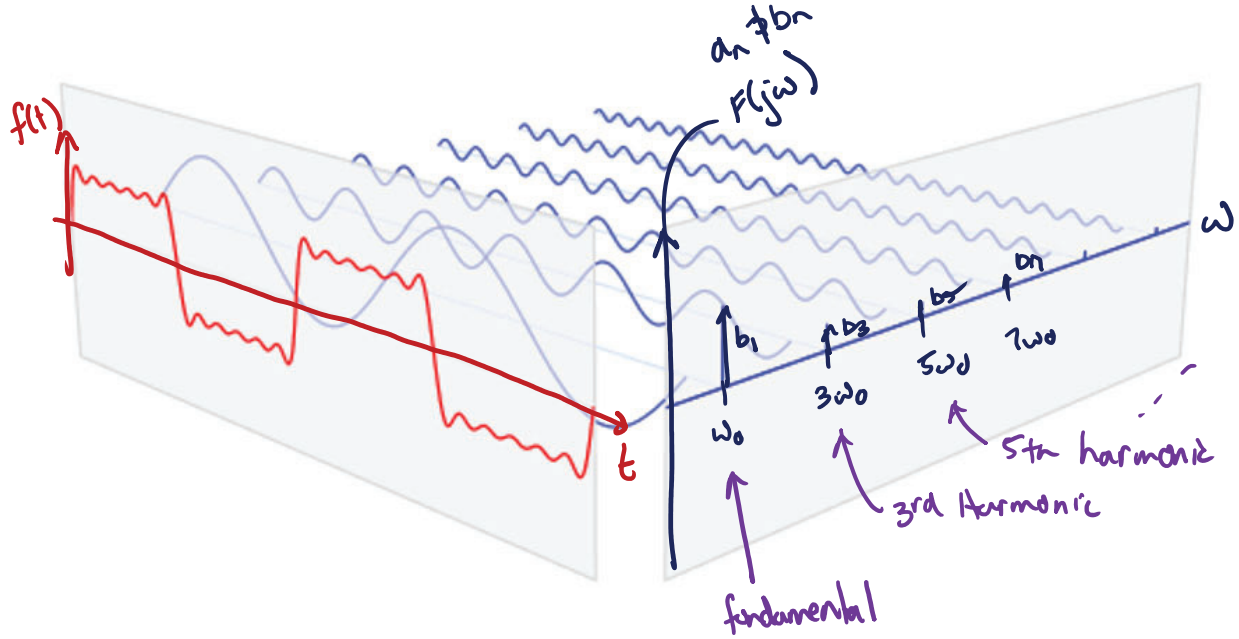
output is:

$$V_o = V_I H(j\omega) = |V_I||H| \angle (\angle V_I + \angle H)$$

Magnitudes multiply

phases add

Frequency Domain



Fourier Series Forms

Assume we have some function $f(t)$ which is periodic with period $T_0 = \frac{2\pi}{\omega_0}$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$f(t)$ can be expressed this way if
1. $f(t)$ is single-valued
2. $\int_{t_0}^{t_0+T_0} |f(t)|\, dt$ exists
3. $f(t)$ had finite discontinuities and max/min per period

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt \qquad a_k = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(k\omega_0 t)\, dt \qquad b_k = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(k\omega_0 t)\, dt$$

Alternate forms

$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \varphi_k) \quad \begin{cases} A_k = \sqrt{{a_k}^2 + {b_k}^2} \\ \varphi_k = \tan^{-1}\left(\frac{b_k}{a_k}\right) \end{cases}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \begin{cases} c_k = \frac{1}{2}(a_k - jb_k) \\ c_{-k} = \frac{1}{2}(a_k + jb_k) \\ c_0 = a_0 \end{cases}$$

complex conjugate

$$c_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jk\omega_0 t}dt$$

Non-periodic Waveforms: Fourier Transform

Fourier series → only periodic signals

What about non-periodic signals?

Let's say our non-periodic signal is actually periodic with $T_0 \to \infty$, $T_0 = \frac{2\pi}{\omega_0}$

Fourier Series: $c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jk\omega_0 t}\, dt$

Fourier Transform: $Tc_k = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = F(\omega)$

Fourier Series: $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

Fourier Transform: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{j\omega t}\, d\omega$

Chapter 14, Lectures 22-33

LAPLACE TRANSFORM

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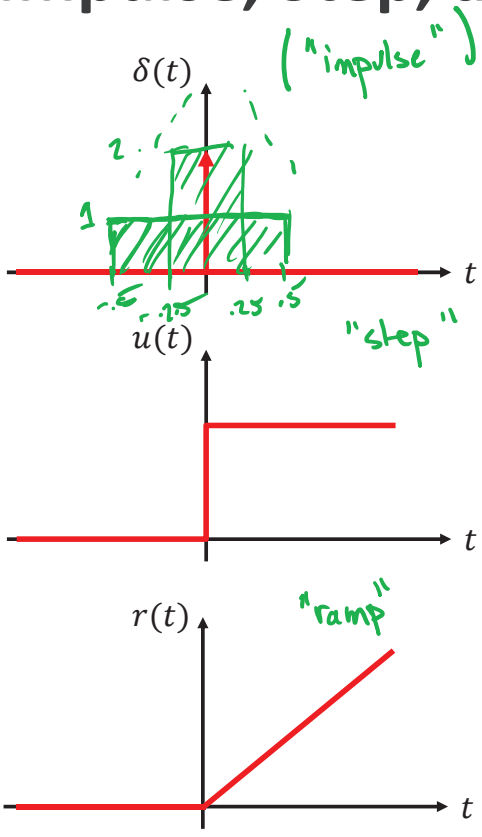
Impulse, Step, and Ramp Functions



$$\delta(t)\begin{cases}0 & t \neq 0\\ \infty & t = 0\end{cases}$$

$$\int_{-\infty}^{+\infty}\delta(t)dt = \int_{0^-}^{0^+}\delta(t)dt = 1$$

d/dt $\quad \int_{-\infty}^{t} f(\tau)d\tau$

$$u(t)\begin{cases}0 & t < 0\\ 1 & t > 0\end{cases}$$

d/dt

$$r(t) = tu(t) = \begin{cases}0 & t \leq 0\\ t & t \geq 0\end{cases}$$

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Sifting Property of Impulse Function $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(\phi)$$

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

"Sifting Property" of $\delta(t)$

Behaves somewhat like a sampler. (more later)

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The Laplace Transform

Take Fourier Transform & replace $(j\omega)$ with $s = (\sigma + j\omega)$

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\,dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} F(s)\,e^{st}\,ds$$

Usually, we'll use the one-sided (unilateral) Laplace Transform

$$\boxed{F(s) = \int_{0^-}^{\infty} e^{-st} f(t)\,dt}$$

Laplace Transform

$$\boxed{f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} F(s)\,e^{st}\,ds}$$

σ_0 = any real #

Inverse Laplace Transform

Valid for signals $f(t) = 0$ for $t < 0$

Short hand notation

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)\}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s)\}$$

Time Domain		Frequency / Fourier Domain		Laplace / s / Complex Freq. Domain	
$f(t)$	ODE	$F(\omega)$	$H(j\omega)$	$F(s)$	$H(s)$
signals	circuits/systems	signals	systems	signals	systems

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Example Signal Laplace Transforms

ex $f(t) = u(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} e^{-st} u(t)\,dt$$

$$= \int_{0^-}^{\infty} 1 \cdot e^{-st}\,dt = \left[\frac{-1}{s} e^{-st}\right]\Big|_{t=0^-}^{t=\infty} = 0 - \left(\frac{-1}{s}\right)$$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}} \quad \text{if } \mathrm{Re}\{s\} > 0 \quad \text{(Region of Convergence)}$$

ex $f(t) = e^{-at} u(t)$

$$\mathcal{L}\{e^{-at} u(t)\} = \int_{0^-}^{\infty} e^{-at} e^{-st}\,dt = \int_{0^-}^{\infty} e^{-(s+a)t}\,dt$$

$$\boxed{\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a}} \quad \text{if } \mathrm{Re}\{s+a\} > 0 \quad \mathrm{Re}\{s\} > -a$$

Generalize: $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

ex $f(t) = r(t) = tu(t)$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} t e^{-st}\,dt = \left(t \frac{-1}{s} e^{-st}\right)\Big|_{t=0^-}^{t=\infty} - \int_{0^-}^{\infty} (1)\frac{-1}{s} e^{-st}\,dt$$

$= \phi$

$u \uparrow \quad \uparrow \frac{dv}{dt}$

Integration by parts

$$\int_{0^-}^{\infty} \frac{dv}{dt} \cdot u\,dt = (u \cdot v)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v \cdot \frac{du}{dt}\,dt$$

$$= -\int_{0^-}^{\infty} \frac{-1}{s} e^{-st}\,dt = \frac{1}{s}\left[\frac{-1}{s} e^{-st}\right]\Big|_{0^-}^{\infty}$$

$$= \frac{1}{s}\left[\frac{-1}{s}(0 - 1)\right] = \boxed{\frac{1}{s^2} = \mathcal{L}\{r(t)\}}$$

$\mathrm{Re}\{s\} > \phi$

TABLE 14.1 Laplace Transform Pairs

$f(t) = \mathcal{L}^{-1}\{\mathbf{F}(s)\}$	$\mathbf{F}(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{\mathbf{F}(s)\}$	$\mathbf{F}(s) = \mathcal{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})\,u(t)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$u(t)$	$\frac{1}{s}$	$\sin \omega t\, u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos \omega t\, u(t)$	$\frac{s}{s^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t), n = 1, 2, \ldots$	$\frac{1}{s^n}$	$\sin(\omega t + \theta)\, u(t)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\cos(\omega t + \theta)\, u(t)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$	$e^{-\alpha t} \sin \omega t\, u(t)$	$\frac{\omega}{(s+\alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t), n = 1,2,\ldots$	$\frac{1}{(s+\alpha)^n}$	$e^{-\alpha t} \cos \omega t\, u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$

TABLE 14.2 Laplace Transform Operations

Operation	$f(t)$	$\mathbf{F(s)}$
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(\mathbf{s}) \pm \mathbf{F}_2(\mathbf{s})$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(\mathbf{s})$
Time differentiation	$\frac{df}{dt}$	$\mathbf{sF(s)} - f(0^-)$
	$\frac{d^2f}{dt^2}$	$\mathbf{s}^2\mathbf{F(s)} - \mathbf{s}f(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$\mathbf{s}^3\mathbf{F(s)} - \mathbf{s}^2f(0^-) - \mathbf{s}f'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^{t} f(t)\,dt$	$\frac{1}{\mathbf{s}}\mathbf{F(s)}$
	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{1}{\mathbf{s}}F(\mathbf{s}) + \frac{1}{\mathbf{s}}\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(\mathbf{s})\mathbf{F}_2(\mathbf{s})$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-as}\mathbf{F(s)}$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F(s+a)}$
Frequency differentiation	$tf(t)$	$-\frac{d\mathbf{F(s)}}{d\mathbf{s}}$
Frequency integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} \mathbf{F(s)}\,d\mathbf{s}$
Scaling	$f(at),\ a \geq 0$	$\frac{1}{a}\mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$
Initial value	$f(0^+)$	$\lim_{\mathbf{s}\to\infty} \mathbf{sF(s)}$
Final value	$f(\infty)$	$\lim_{\mathbf{s}\to 0} \mathbf{sF(s)}$, all poles of $\mathbf{sF(s)}$ in LHP
Time periodicity	$f(t) = f(t+nT)$, $n = 1, 2, \ldots$	$\frac{1}{1-e^{-T_s}}\mathbf{F}_1(\mathbf{s})$, where $\mathbf{F}_1(\mathbf{s}) = \int_{0^-}^{T} f(t)\,e^{-\mathbf{s}t}\,dt$

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Initial and Final Value Theorems

Initial Value Theorem:

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \lim_{s\to\infty}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt\right] = \lim_{s\to\infty}\left[sF(s) - f(0^-)\right]$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} e^{-st}\frac{df}{dt}\,dt + \int_{0^+}^{\infty} e^{-st}\frac{df}{dt}\,dt\right] = \lim_{s\to\infty}\left[sF(s)\right] - f(0^-)$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} \frac{df}{dt}\,dt\right] = \lim_{s\to\infty}\left[sF(s)\right] - f(0^-)$$

$$f(0^+) - f(0^-) = \lim_{s\to\infty}\left[sF(s)\right] - f(0^-)$$

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)}$$

fast response ← Really high frequencies

Final Value Theorem:

$$\lim_{s\to 0}\left[\int_{0^-}^{\infty} e^{-st}\frac{df}{dt}\,dt\right] = \lim_{s\to 0}\left[sF(s) - f(0^-)\right]$$

$$\lim_{t\to\infty} f(t) - f(0^-) = \lim_{s\to 0}\left[sF(s)\right] - f(0^-)$$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sF(s)\right]}$$

slow response → ← low frequencies

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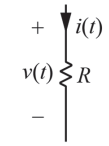
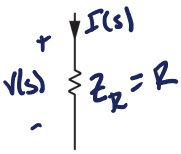
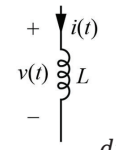
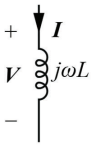
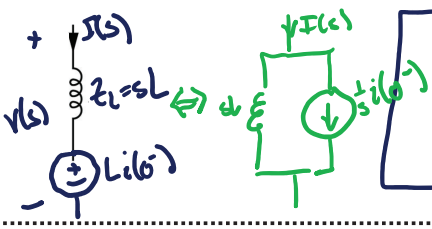
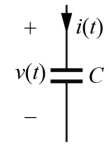
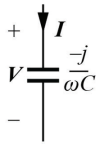
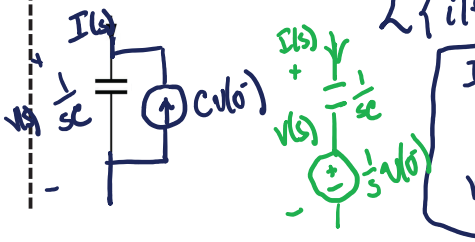
Circuit Laplace Transform

Time Domain	Phasor Domain	s-Domain
 $v(t) = i(t)R$	$V = IR$	
 $v(t) = L\frac{di}{dt}$	 $V = j\omega LI$	
 $i(t) = C\frac{dv}{dt}$	 $V = \frac{-j}{\omega C}I$	

$\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)R\}$

$V(s) = I(s) \cdot R$

$Z_R = R$

still called "impedance"

$\mathcal{L}\{v(t)\} = \mathcal{L}\{L\frac{di}{dt}\}$

$V(s) = L[sI(s) - i(0^-)]$

or

$I(s) = \frac{1}{sL}V(s) + \frac{1}{s}i(0^-)$

$Z_L = sL$

$\mathcal{L}\{i(t)\} = \mathcal{L}\{C\frac{dv}{dt}\}$

$I(s) = CsV(s) - Cv(0^-)$

or

$V(s) = \frac{1}{sC}I(s) + \frac{1}{s}v(0^-)$

$Z_C = \frac{1}{sC}$

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Inverse Transforms

Transfer Function $\frac{V_o(s)}{V_I(s)} = H(s)$ → $V_o(s) = H(s) \cdot V_I(s)$

$v_o(t) = \mathcal{L}^{-1}\{H(s) \cdot V_I(s)\}$

$H(s)$: from circuit manipulation; $V_I(s)$: from Laplace Transform of $v_i(t)$

We will find $V_o(s)$ is usually of ratio of polynomials in s that needs to be inverse transformed

Generally, this will require factoring polynomials

ex $F(s) = \frac{10}{s^2+4s+4} = \frac{10}{(s+2)(s+2)} = \frac{10}{(s+2)^2}$

roots of $as^2 + bs + c$ $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $r = -2$

$f(t) = \mathcal{L}^{-1}\left\{\frac{10}{(s+2)^2}\right\} = 10\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = 10te^{-2t}u(t) = f(t)$

ex $F(s) = \frac{5s+1}{s+1}$ order of numerator = order of denominator → use long division

$s+1 \overline{)\,5s+1}$

$-5s+5$

-4

$F(s) = 5 + \frac{-4}{s+1}$

$f(t) = 5\delta(t) - 4e^{-t}u(t)$

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Partial Fraction Expansion / Decomposition

$$\frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i} = \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)} = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \cdots \frac{k_N}{(s-p_N)}$$

factoring (M) PFE — k_1, k_2 called "residues"

First, assume all p_i are unique & $M < N$ (will do alternate case next)

PFE: to find k_i, multiply both sides by $(s-p_i)$ & evaluate at $s=p_i$

ex for k_2:

$$\left.\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}\right| = \left.\left[\frac{k_1(s-p_2)}{(s-p_1)} + \frac{k_2(s-p_2)}{(s-p_2)} + \cdots \frac{k_N(s-p_2)}{(s-p_N)}\right]\right|_{s=p_2} = k_2$$

sometimes called "cover-up" method

ex $F(s) = \frac{4(s+2)}{s^2+4s+3} = \frac{4(s+2)}{(s+3)(s+1)} = \frac{k_1}{s+3} + \frac{k_2}{s+1} = \frac{2}{s+3} + \frac{2}{s+1}$

$$k_1 = \left.\frac{4(s+2)}{(s+1)}\right|_{s=-3} = 2$$

$$k_2 = \left.\frac{4(s+2)}{(s+3)}\right|_{s=-1} = 2$$

$f(t) = \mathcal{L}^{-1}\{F(s)\} =$

$$\boxed{f(t) = 2e^{-3t}u(t) + 2e^{-t}u(t)}$$

PFE: Repeated Roots

$$F(s) = \frac{N(s)}{(s-p_1)(s-p_2)^2} = \frac{k_1}{s-p_1} + \frac{k_2}{(s-p_2)} + \frac{k_3}{(s-p_2)^2} = \frac{k_1}{s-p_1} + \frac{k_2 s + (k_3 - p_2 k_2)}{(s-p_2)^2}$$

two poles @ $s=p_2$

k_1 → no change, do cover up method

for k_2 & k_3 either { (1) equating coefficients (2) differentiating method

Equating Coefficients

ex $F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2} = \frac{k_1}{s+2} + \frac{As+B}{(s+10)^2} = \frac{1}{s+2} + \frac{31s-50}{(s+10)^2}$

$$= \frac{1}{s+2} + \frac{31}{s+10} + \frac{-360}{(s+10)^2}$$

$$k_1 = \left.\frac{32s(s+1)}{(s+10)^2}\right|_{s=-2} = 1$$

for A & B multiply both sides by full original denominator & equate coefficients of s

$$32s(s+1) = k_1(s+10)^2 + (As+B)(s+2)$$

$$32s^2 + 32s = s^2 + 20s + 100 + As^2 + 2As + Bs + 2B$$

s^2: $32 = 1 + A \rightarrow A = 31$ ✓

s: $32 = 20 + 2A + B \rightarrow B = -50$ ✓

1: $0 = 100 + 2B$

$$\boxed{f(t) = \left[e^{-2t} + 31e^{-10t} - 360te^{-10t}\right]u(t)}$$

Repeated Roots: Differentiation

Method 2: Differentiating

ex $F(s) = \frac{32s(s+1)}{(s+2)(s+10)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+10} + \frac{k_3}{(s+10)^2}$

k_1 & k_3 → normal cover-up method

$$k_1 = \left.\frac{32s(s+1)}{(s+10)^2}\right|_{s=-2} = 1$$

$$k_3 = \left.\frac{32s(s+1)}{(s+2)}\right|_{s=-10} = -360$$

for k_2: multiply both sides by $(s+10)^2$, then differentiate w.r.t. s before plugging in $s=-10$

$$\left[\frac{d}{ds}\left[\frac{32s(s+1)}{s+2}\right]\right]\Big|_{s=-10} = \left[\frac{d}{ds}\left[\frac{g(s)}{k_1}(s+10)^2 + \frac{k_2}{s+10}(s+10)^2 + k_3\right]\right]\Big|_{s=-10}$$

$$\left[g(s)\,2(s+10) + g'(s)(s+10)^2 + k_2 + 0\right]_{s=-10}$$

$$k_2 = \left[\frac{d}{ds}\left[\frac{32s^2+32s}{s+2}\right]\right]\Big|_{s=-10} = \left.\frac{(64s+32)(s+2)-(32s^2+32s)}{(s+2)^2}\right|_{s=-10}$$

$$= 31$$

$$F(s) = \frac{1}{s+2} + \frac{31}{s+10} + \frac{-360}{(s+10)^2}$$

PFE: Complex Roots

ex $F(s) = \frac{1}{s^2-2s+2}$

roots @ $\frac{2 \pm \sqrt{4-4\cdot1\cdot2}}{2} = 1 \pm j$

Note: for real circuits & signals, complex roots (and their residues) will always show up in conjugate pairs

$$F(s) = \frac{1}{s^2-2s+2} = \frac{1}{(s-(1+j))(s-(1-j))} = \frac{k_1}{s-(1+j)} + \frac{k_1^*}{s-(1-j)}$$

Option 1: Do nothing Different

$$k_1 = \left.\frac{1}{s-(1-j)}\right|_{s=1+j} = \frac{1}{2j} = \frac{-j}{2}$$

$$k_2 = \left.\frac{1}{s-(1+j)}\right|_{s=1-j} = \frac{1}{-2j} = \frac{j}{2} = k_1^*$$

(could skip knowing $k_2 = k_1^*$)

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{j}{2}\mathcal{L}^{-1}\left\{\frac{-1}{s-(1+j)} + \frac{1}{s-(1-j)}\right\} =$$

$$f(t) = \frac{j}{4}\left(e^{(1+j)t}u(t) + e^{(1-j)t}u(t)\right)$$

$$= \frac{j}{2}e^{t}\left(e^{jt} + e^{-jt}\right)u(t) = e^{t}\frac{\left(e^{jt}-e^{-jt}\right)}{2j}u(t)$$

$$= e^{t}\sin(t)u(t)$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

Option 2: Let's do the general case:

$$= \mathcal{L}^{-1}\left\{\frac{k}{s-(\sigma+j\omega)} + \frac{k^*}{s-(\sigma-j\omega)}\right\}$$

$$= ke^{(\sigma+j\omega)t}u(t) + k^*e^{(\sigma-j\omega)t}u(t)$$

$$= e^{\sigma t}\left(ke^{j\omega t} + k^*e^{-j\omega t}\right)u(t)$$

$$= e^{\sigma t}\left(\mathrm{Re}\{k\}\left(e^{j\omega t}+e^{-j\omega t}\right) + j\,\mathrm{Im}\{k\}\left(e^{j\omega t}-e^{-j\omega t}\right)\right)u(t)$$

$$= e^{\sigma t}\left(\mathrm{Re}\{k\}\,2\cos(\omega t) - \mathrm{Im}\{k\}\,2\sin(\omega t)\right)u(t)$$

$$= 2e^{\sigma t}\left(\sqrt{\mathrm{Re}\{k\}^2+\mathrm{Im}\{k\}^2}\cos\left(\omega t - \tan^{-1}\left(\frac{\mathrm{Im}\{k\}}{\mathrm{Re}\{k\}}\right)\right)\right)u(t)$$

$$= 2e^{\sigma t}\left(|k|\cos(\omega t - \angle k)\right)u(t)$$

$$\frac{k}{s-(\sigma+j\omega)} + \frac{k^*}{s-(\sigma-j\omega)} = 2e^{\sigma t}|k|\cos(\omega t - \angle k)u(t)$$

Option 3: Manipulate polynomial into one of the existing items in the table

Complex Roots: Option 3

ex $F(s) = \frac{2}{s^2 - 2s + 2}$

roots @ $\frac{2 \pm \sqrt{4 - 4(1)2}}{2 \cdot 1} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2j}{2} = 1 \pm j$

some exponential, some sinusoid

$= \frac{k}{(s+\alpha)^2 + \omega^2}$

$(s-1)^2 = s^2 - 2s + 1$

$= \frac{2}{(s-1)^2 + 1} = 2\left(\frac{1}{(s-1)^2 + 1}\right)$

$\alpha = -1$, $\omega = \sqrt{1} = 1$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2e^{t}\sin(t)$$

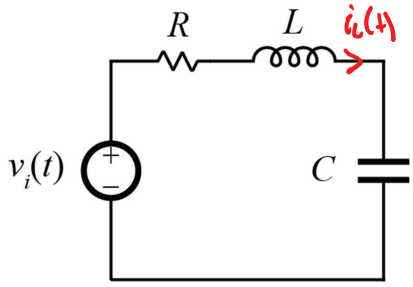
Example

find $v_o(t)$ for:

$$v_i(t) = \sin(2t)\,u(t)$$

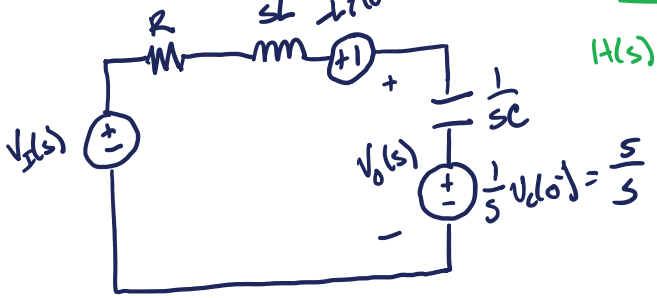
$$L = 500\text{mH},\quad C = 500\mu\text{F},\quad R = 2\Omega$$

$$v_c(0) = 5\text{V},\quad i_L(0) = -2\text{A}$$



Source: $V_I(s) = \mathcal{L}\{v_i(t)\} = \frac{2}{s^2+4}$

Circuit:

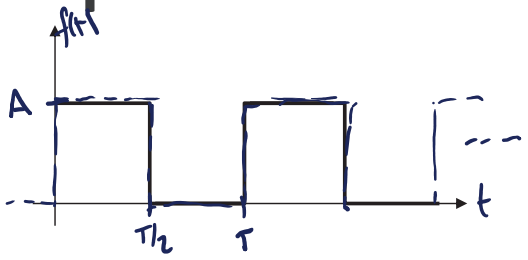


Transfer Function:

$$H(s) = \left.\frac{V_o(s)}{V_I(s)}\right|_{\text{all other indep sources are shut off}} = \frac{\frac{1}{sC}}{\frac{1}{sC} + sL + R} = \frac{1}{s^2LC + sCR + 1}$$

$$H(s) = \frac{1}{\frac{s^2}{4} + s + 1} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

Laplace Transform of Periodic Signals



$$f(t) = A u(t) - A u(t - T/2) + A u(t - T) - A u(t - T - T/2) + A u(t - 2T) - A u(t - 2T - T/2) + \cdots$$

$$f(t) = \sum_{k=0}^{\infty} A u(t - kT) - A u(t - T/2 - kT)$$

$$F(s) = \mathcal{L}\{f(t)\} = \sum_{k=0}^{\infty} A\frac{1}{s}e^{-kTs} - A\frac{1}{s}e^{(-T/2+kT)s}$$

$$= A\frac{1}{s}\left(1 - e^{-\frac{T}{2}s}\right)\sum_{k=0}^{\infty} e^{-kTs}$$

$$F(s) = \frac{A\frac{1}{s}\left(1 - e^{-\frac{T}{2}s}\right)}{1 - e^{-sT}}$$

Recall Geometric series: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r| < 1$

$$\sum_{k=0}^{\infty} \left(e^{-sT}\right)^k$$

$$F(s) = \frac{F_1(s)}{1 - e^{-sT}}$$ ← Laplace TF of 1st period

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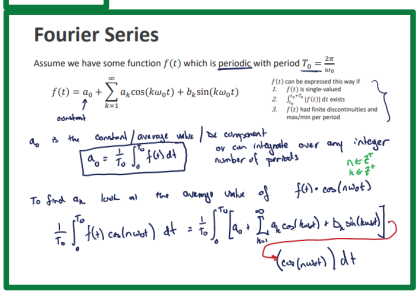
Laplace Explanation

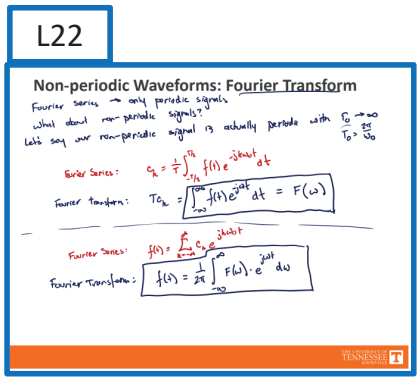
$$F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\, dt = \int_{0^-}^{+\infty} e^{-\sigma t} e^{-j\omega t} f(t) dt$$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\sigma t} e^{j\omega t} F(s)\, ds$$

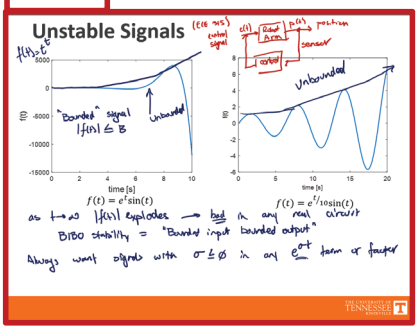
L20



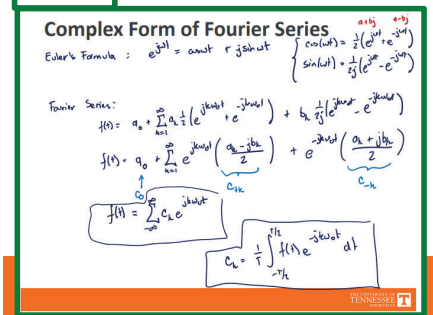
L22



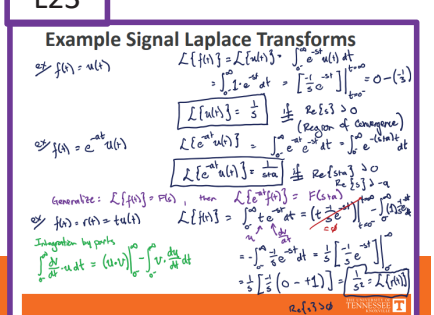
L28



L21



L23

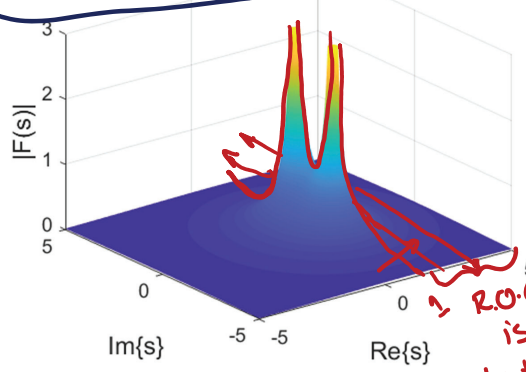


The s-plane

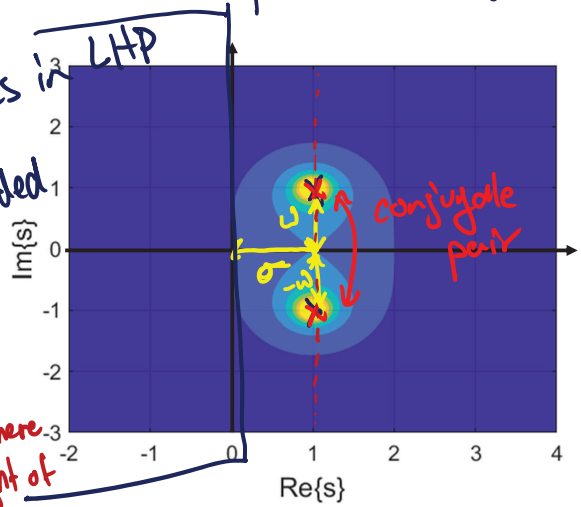
If all poles are in the open Left half-plane, the signal is bounded and goes to zero for $t \to \infty$

$\sigma = 1 \quad \omega = 1$

poles @ $\sigma \pm j\omega$



Poles in LHP = bounded



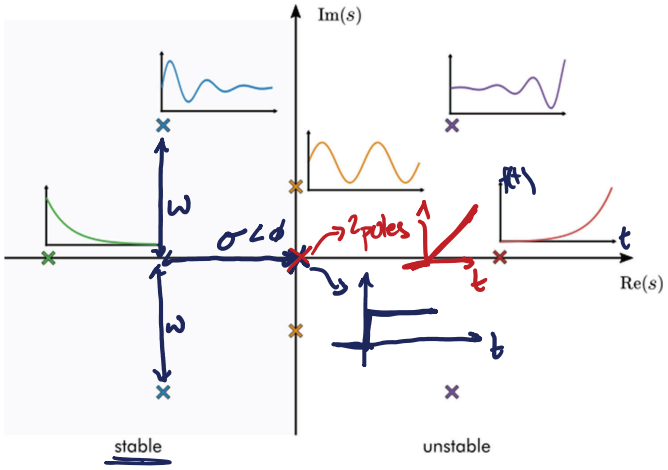
→ R.O.C. is anywhere to the right of all poles

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-(1+j))(s-(1-j))} \rightarrow \text{poles @ } s = (1 \pm j)$$

$f(t) = e^{t} \sin t \, u(t)$

$= e^{\sigma t} \sin(\omega t) u(t)$

Pole-Zero Plot



Pole-zero plot

x = pole

o = zero

(1) If a pole is complex, its conjugate is also a pole for any real signal/system.
Same for zeros

(2) If all poles are in the open LHP, the signal is bounded or the system is BIBO stable
"Bounded input, Bounded output"
with $\lim_{t \to \infty} f(t) = 0$

RHP poles = unstable = bad!

RHP zeros → not good, but can still be stable

If $F(s)$ has poles @ $\sigma \pm j\omega$

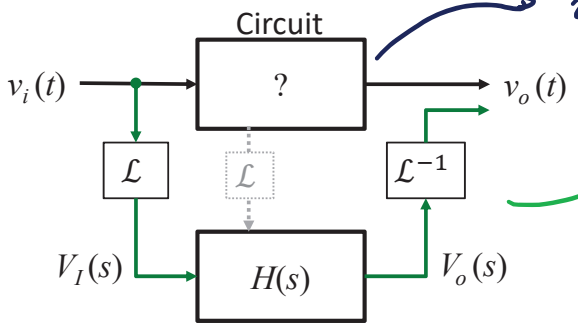
$$F(s) = \frac{N(s)}{\cdots (s-(\sigma+j\omega))(s-(\sigma-j\omega)) \cdots}$$

by PFE $\quad F(s) = \cdots \frac{k}{s-(\sigma+j\omega)} + \frac{k^*}{s-(\sigma-j\omega)} + \cdots$

$f(t) = \cdots + 2|k| e^{\sigma t} \cos(\omega t - \angle k) + \cdots$

https://www.mathworks.com/matlabcentral/fileexchange/94635-transfer-function-analysis-of-dynamic-systems

System I/O Relationship



201 type analysis: solve Diff EQ w/ ICs for $v_o(t)$

202 analysis in s-domain

$H(s) = \mathcal{L}\{\text{circuit}\}$

$V_I(s) = \mathcal{L}\{v_i(t)\}$

$\Big\} \; v_o(t) = \mathcal{L}^{-1}\{H(s)V_I(s)\}$

$v_o(t) = \mathcal{L}^{-1}\{V_I(s)\cdot H(s)\} = V_o(s)$

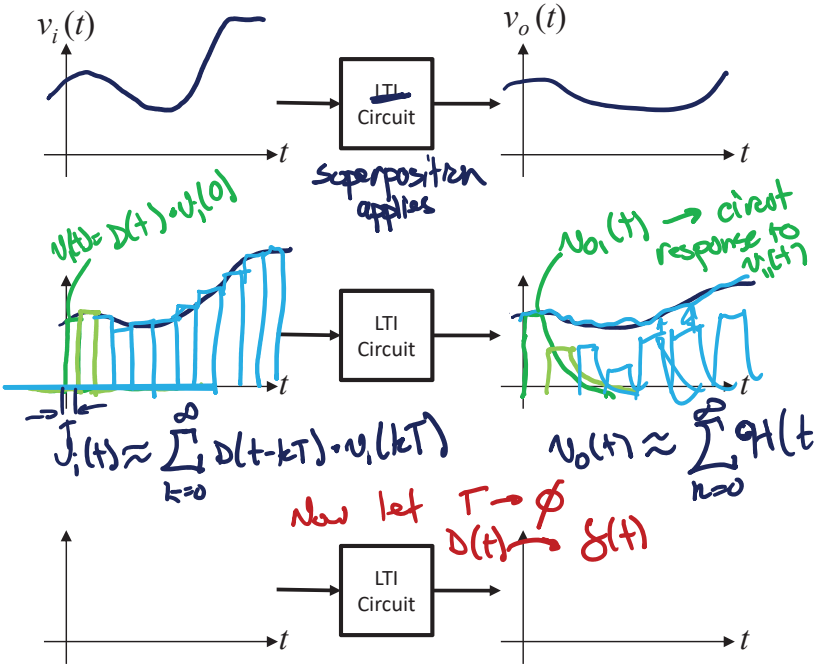
what is the time-domain equivalent of multiplication in the Laplace domain

$v_i(t) = \mathcal{L}^{-1}\{V_I(s)\}$

$h(t) = \mathcal{L}^{-1}\{H(s)\}$ = ouput when the input is $\delta(t)$

$\mathcal{L}^{-1}\{H(s)\} = h(t) =$ "impulse response"

Convolution



$v_i(t) = \int_0^{\infty} \delta(t-\tau)\, v_i(\tau)\, d\tau$

by sifting property, yes

convolution integral

$v_o(t) = \int_0^{\infty} h(t-\tau)\, v_i(\tau)\, d\tau$

or

$v_o(t) = \int_0^{\infty} v_i(t-\tau)\, h(\tau)\, d\tau$

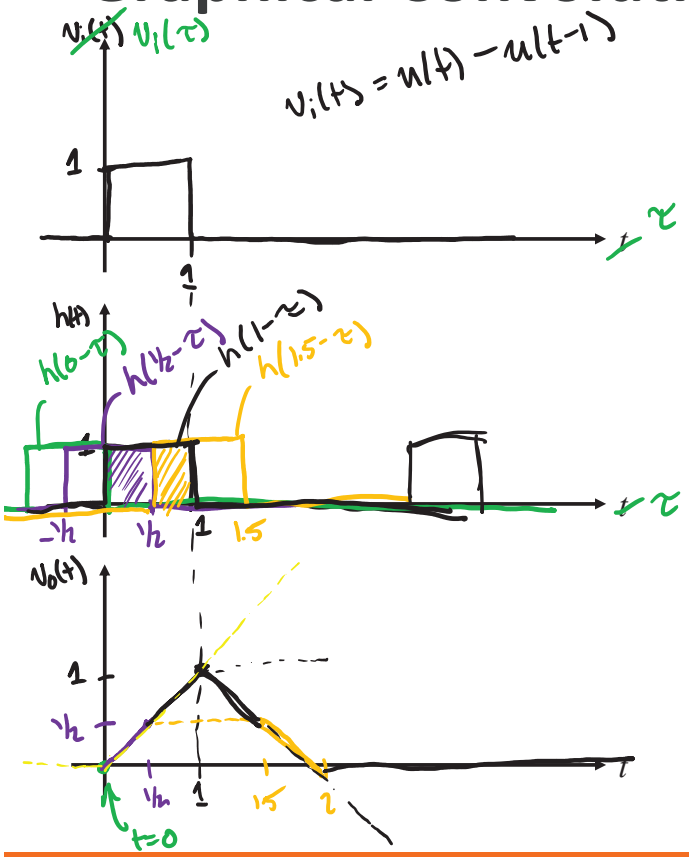
formally, integral limits should go $-\infty$ to ∞

But $v_i(t)$ is zero for $t<0$

& for causal systems $h(t)$ is zero for $t<0$

causal = can't predict the future

Graphical Convolution



$$v_o(t) = \int_0^\infty h(t-\tau)\, v_i(\tau)\, d\tau$$

$$v_o(t) = \begin{cases} 0, & t < 0 \\ \int_0^t 1\,dt = t, & 0 \le t < 1 \\ \int_{t-1}^1 1\,dt = 2-t, & 1 \le t < 2 \\ \phi, & 2 < t \end{cases}$$

$$v_o(t) = r(t) - 2r(t-1) + r(t-2)$$

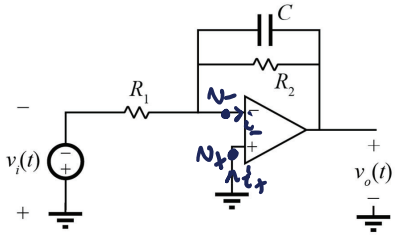
$$= t\,u(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

$$V_I(s) = \frac{1}{s} - e^{-s}\frac{1}{s} = H(s)$$

$$V_o(s) = V_I(s)\,H(s) = \frac{1}{s^2} + e^{-2s}\frac{1}{s^2} - 2e^{-s}\frac{1}{s^2}$$

$$v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = t\,u(t) + (t-2)u(t-2) - 2(t-1)u(t-1)$$

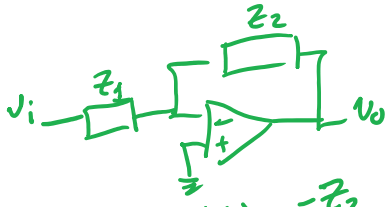
Example Problem



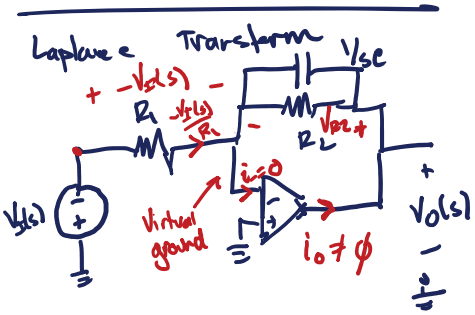
Ideal opamp assumptions

(1) If there is negative feedback $v_+ = v_-$ (virtual short)

(2) $i_+ = i_- = \phi$



$$\frac{V_o(s)}{V_I(s)} = \frac{-Z_2}{Z_1}$$



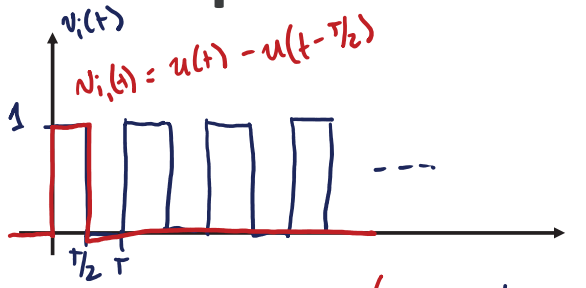
$$I_{R1} = \frac{-V_I(s)}{R_1}$$

$$V_{R2} = -(I_{R1})\left(\frac{1}{sC} \parallel R_2\right)$$

$$V_o = \phi + V_{R2} = \frac{V_I(s)}{R_1}\left(\frac{1}{sC} \parallel R_2\right)$$

$$\boxed{V_o(s) = V_I(s)\,\frac{R_2}{R_1}\,\frac{1}{1+R_2sC}}$$

Example Problem



Laplace transform of 1st period

$$V_{I_1}(s) = \frac{1}{s} - e^{-sT/2}\frac{1}{s}$$

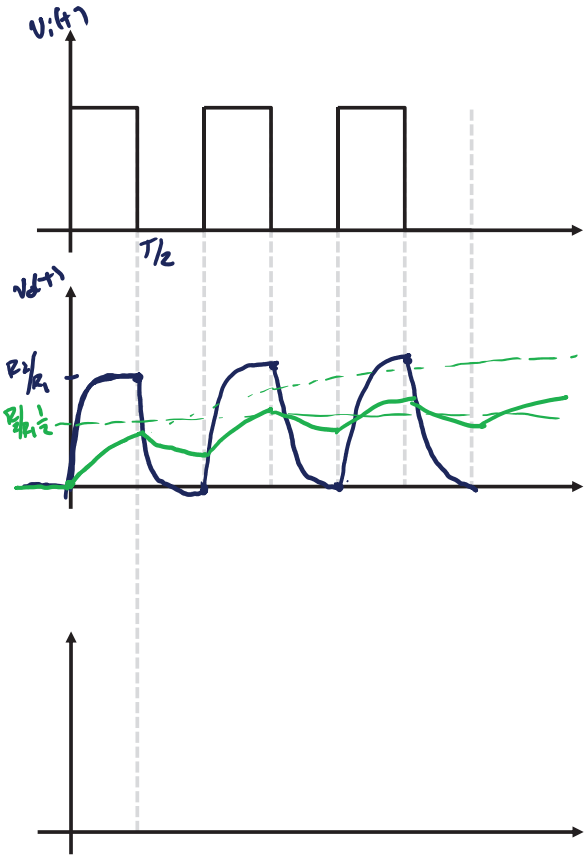
$$V_I(s) = \frac{V_{I_1}(s)}{1-e^{-sT}} = \frac{1}{s}\frac{\left(1-e^{-sT/2}\right)}{1-e^{-sT}}$$

$$V_o(s) = V_I(s)H(s) = \left(\frac{R_2}{R_1}\frac{1}{1+sCR_2}\frac{1}{s}\right)\frac{\left(1-e^{-sT/2}\right)}{\left(1-e^{-sT}\right)}$$

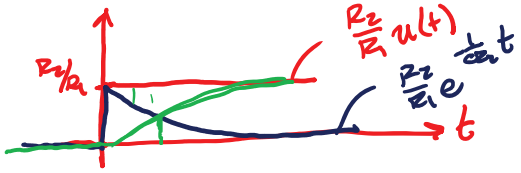
$$\frac{R_2}{R_1}\frac{1}{1+sCR_2}\cdot\frac{1}{s} = \frac{k_1}{s} + \frac{k_2}{1+sCR_2} = \frac{R_2/R_1}{s} + \frac{-CR_2\frac{R_2}{R_1}}{1+sCR_2}$$

$$V_o(s) = \left[\frac{R_2/R_1}{s} - \frac{R_2/R_1}{s+\frac{1}{CR_2}}\right]\frac{\left(1-e^{-sT/2}\right)}{\left(1-e^{-sT}\right)}$$

$$v_o(t) = \sum_{k=0}^{\infty}\left[\frac{R_2}{R_1}u(t+kT) - \frac{R_2}{R_1}e^{-\frac{1}{CR_2}(t+kT)}u(t+kT)\right] - \left[\frac{R_2}{R_1}u(t-T/2+kT) - \frac{R_2}{R_1}e^{-\frac{1}{CR_2}(t-T/2+kT)}u(t-T/2+kT)\right]$$



$$v_o(t) = \sum_{k=0}^{\infty}\frac{R_2}{R_1}\left(1-e^{-\frac{1}{CR_2}(t-kT)}\right)u(t-kT) - \sum_{k=0}^{\infty}\frac{R_2}{R_1}\left(1-e^{-\frac{1}{CR_2}(t-\frac{T}{2}-kT)}\right)u(t-\frac{T}{2}-kT)$$



if $\frac{T}{2} \gg CR_2$ → goes through multiple time constants of the exponential

if $\frac{T}{2} \ll CR_2$ → less than 1 time constant

PART III: BODE

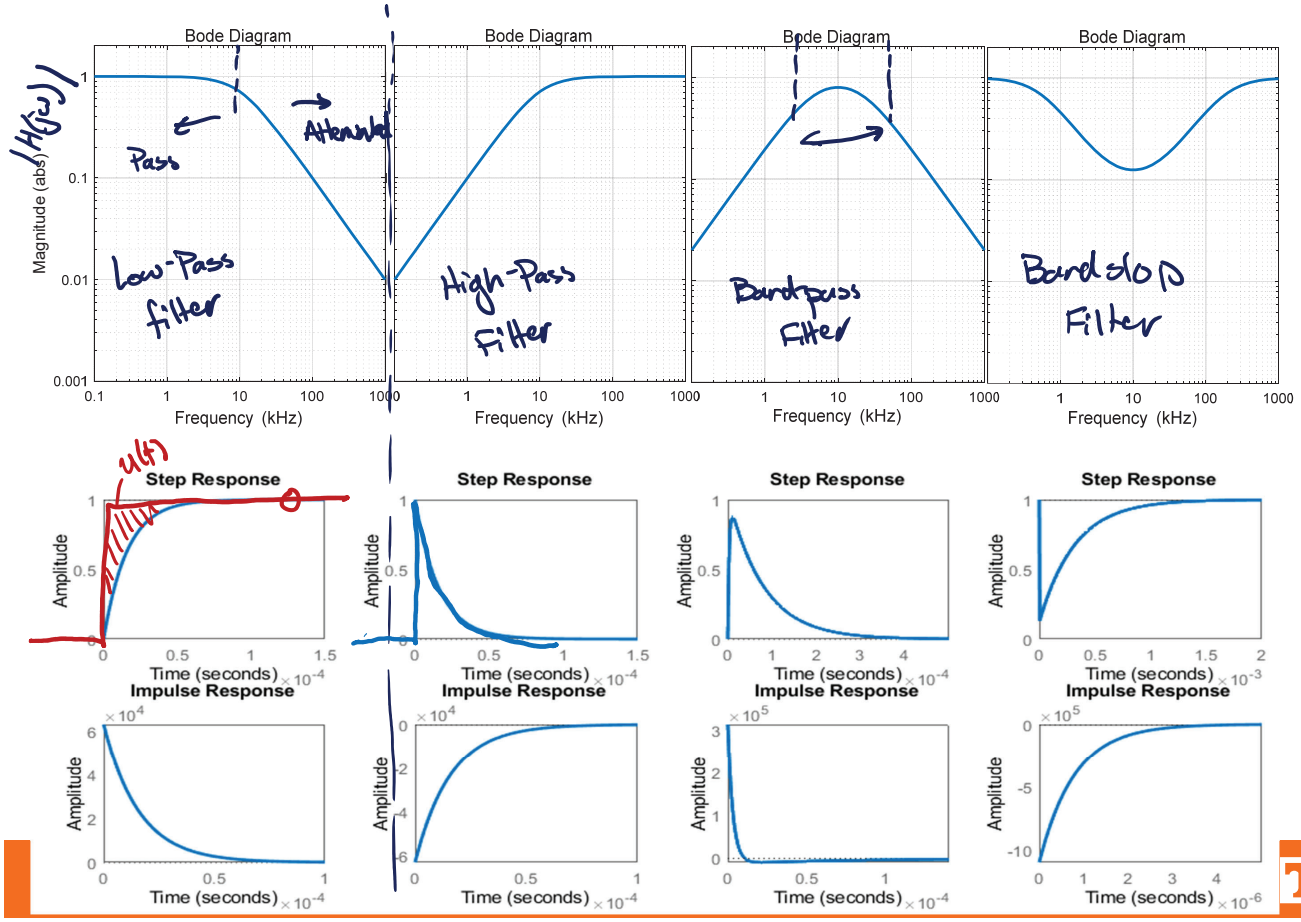
THE UNIVERSITY OF TENNESSEE KNOXVILLE

Chapter 15, Lectures 36-41

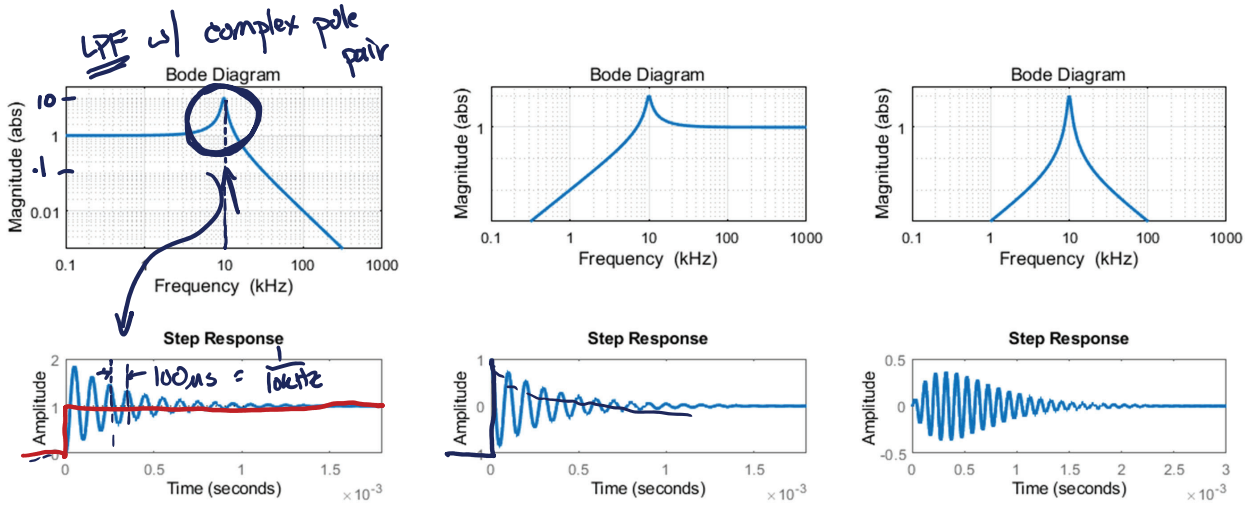
FREQUENCY RESPONSE

THE UNIVERSITY OF TENNESSEE KNOXVILLE

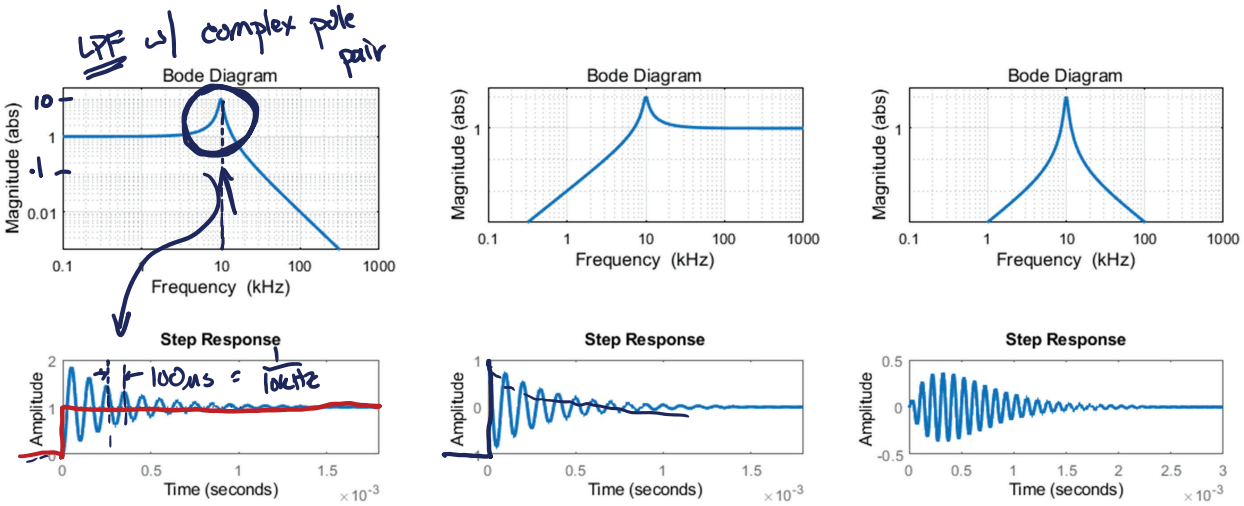
Frequency Response and Circuit Behavior



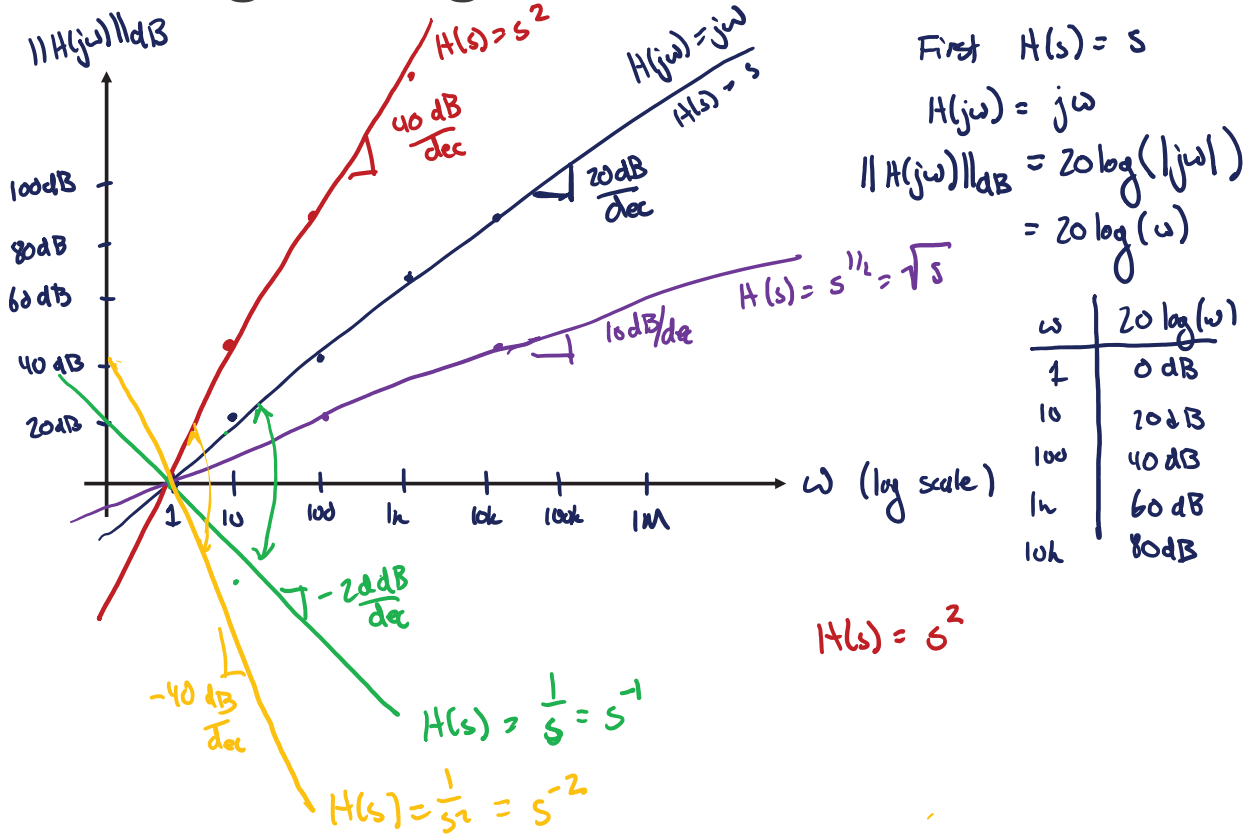
Complex Poles



Complex Poles



Plotting on Logarithmic Axes



First $H(s) = s$

$H(j\omega) = j\omega$

$\|H(j\omega)\|_{dB} = 20\log(|j\omega|)$

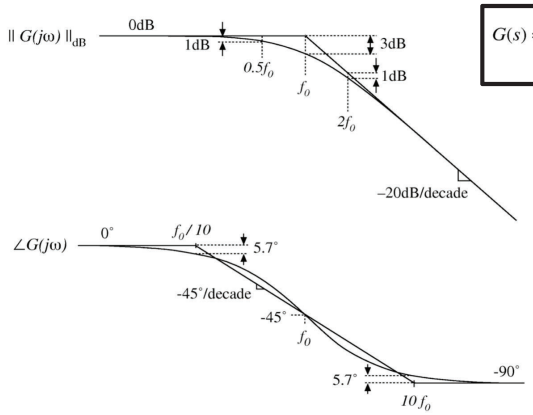
$= 20\log(\omega)$

ω	20 log(ω)
1	0 dB
10	20dB
100	40 dB
1k	60 dB
10k	80dB

$H(s) = s^2$

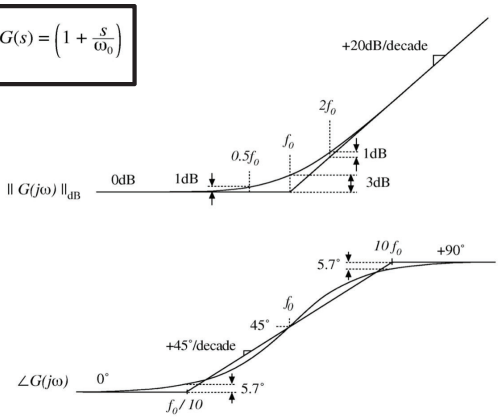
Real Pole

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$



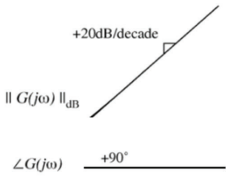
Real Zero

$$G(s) = \left(1 + \frac{s}{\omega_0}\right)$$



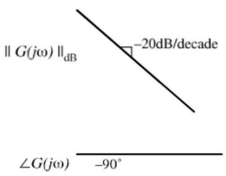
LF Zero

$$G(s) = s$$



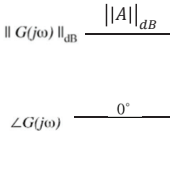
LF Pole

$$G(s) = \frac{1}{s}$$



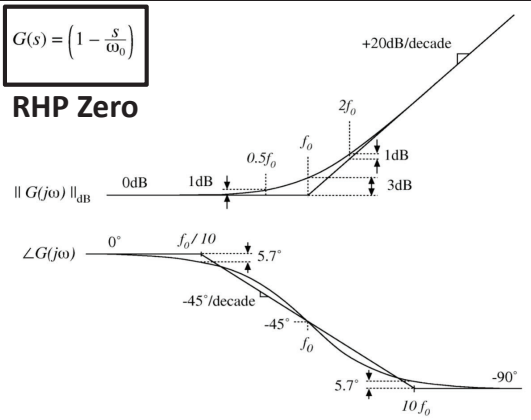
Constant

$$G(s) = A$$



RHP Zero

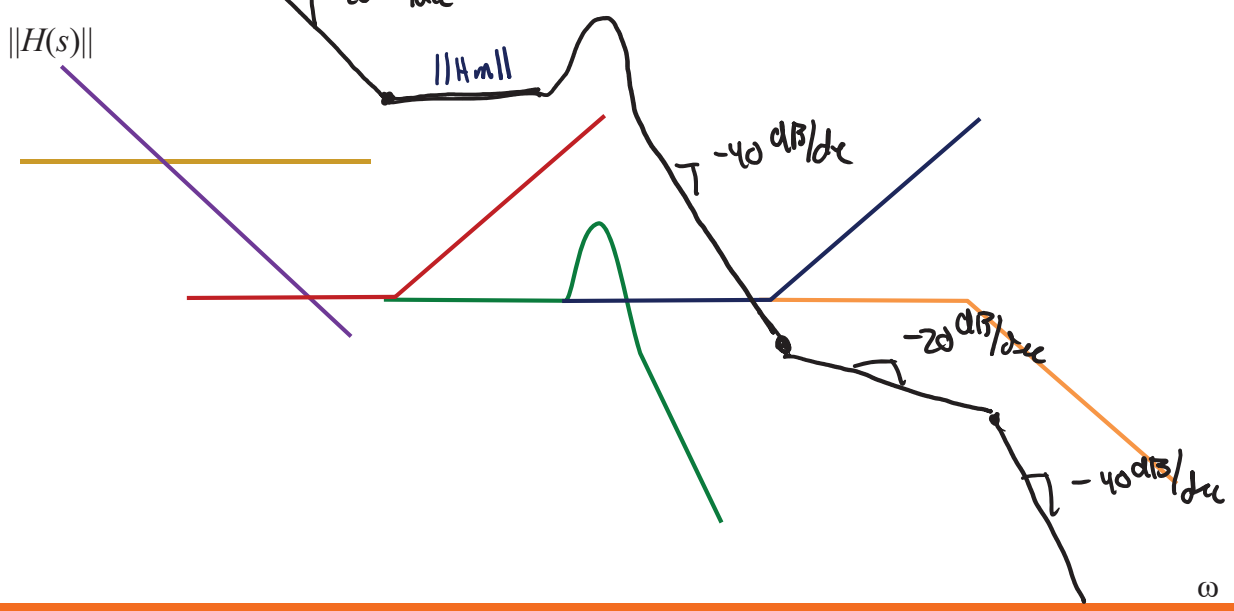
$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$



Example

A	w_{z1}	w_{z2}	ω_0	Q	ω_p
1000	100	10k	1k	10	100k

$$H(s) = A \frac{\left(1 + \frac{s}{w_{z1}}\right)\left(1 + \frac{s}{w_{z2}}\right)}{s\left(\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{Q\omega_0} + 1\right)\left(1 + \frac{s}{\omega_p}\right)}$$



Example

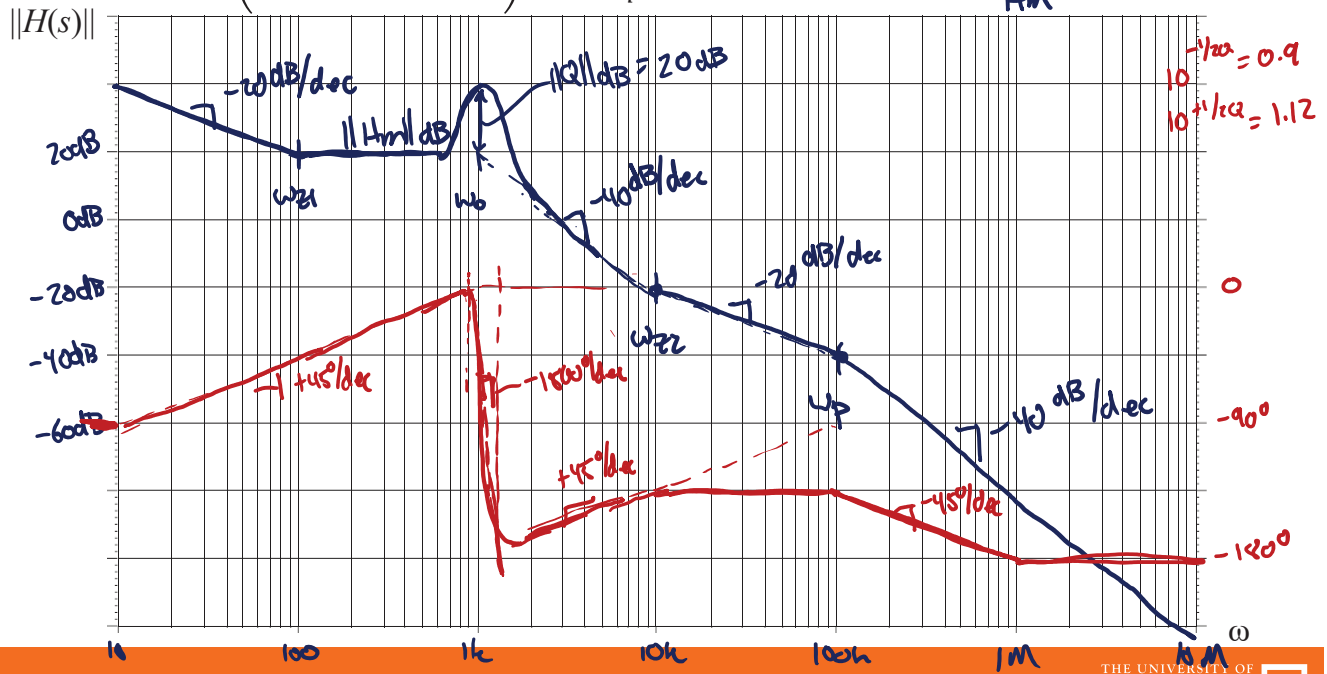
LF: $\frac{A}{s}$

A	w_{z1}	w_{z2}	ω_0	Q	ω_p
1000	100	10k	1k	10	100k

$$H(s) = A\frac{\left(1+\frac{s}{w_{z1}}\right)\left(1+\frac{s}{w_{z2}}\right)}{s\left(\left(\frac{s}{\omega_0}\right)^2+\frac{s}{Q\omega_0}+1\right)\left(1+\frac{s}{\omega_p}\right)}$$

$\frac{A}{s}\left(1+\frac{s}{w_{z1}}\right) = \frac{A}{w_{z1}}\left(\frac{w_{z1}}{s}+1\right)$

↑ $H_m = 0 \Rightarrow 20$ dB



Approximate Graphical Analysis

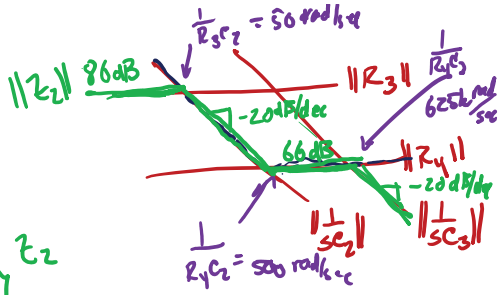
- $\|H_1 \cdot H_2\|_{dB} = \|H_1\|_{dB} + \|H_2\|_{dB}$ — Multiplication (linear) → Addition (log)
- $\|\omega^n\|_{dB} = 20(n)\log(\omega) \Rightarrow 20(n)$ dB/dec line
- $\|1+jx\|_{dB} \simeq \begin{cases} \|x\|_{dB}, & x \gg 1 \\ \frac{1}{\sqrt{2}} = -3dB, & x = 1 \\ 1, & x \ll 1 \end{cases}$

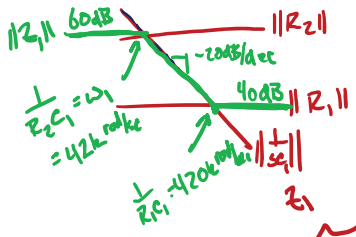
Addition (linear) $\simeq$ max( ) log (+ look at intersection)

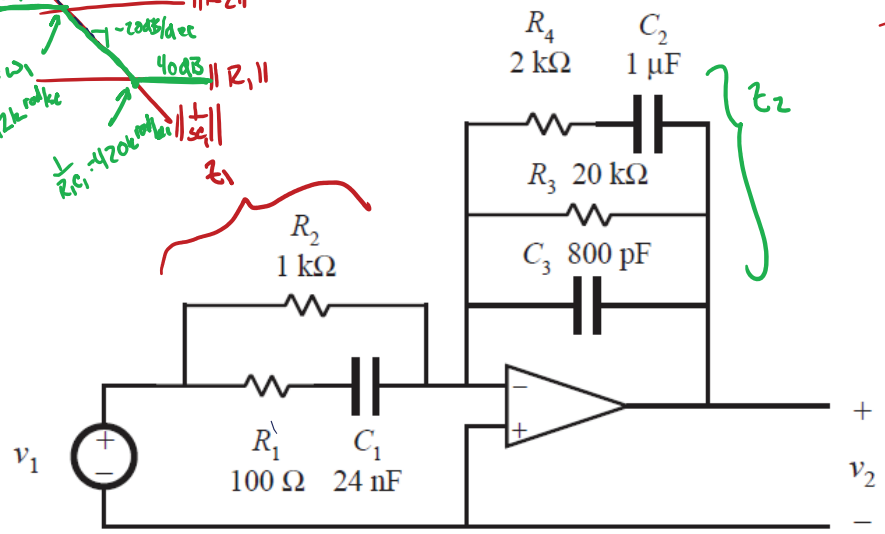
Series $\|z_1 + z_2\|_{dB} = \begin{cases} \|z_1\|_{dB}, & |z_1| \gg |z_2| \\ \text{look closer}, & |z_1| = |z_2| \\ \|z_2\|_{dB}, & |z_1| \ll |z_2| \end{cases}$

Parallel $\|(z_1 \| z_2)\|_{dB} = \left\|\frac{z_1 z_2}{z_1+z_2}\right\|_{dB} \simeq \begin{cases} \|z_2\|_{dB} & |z_1| \gg |z_2| \\ \text{look closer} & |z_1| = |z_2| \\ \|z_1\|_{dB} & |z_1| \ll |z_2| \end{cases}$

Graphical Analysis

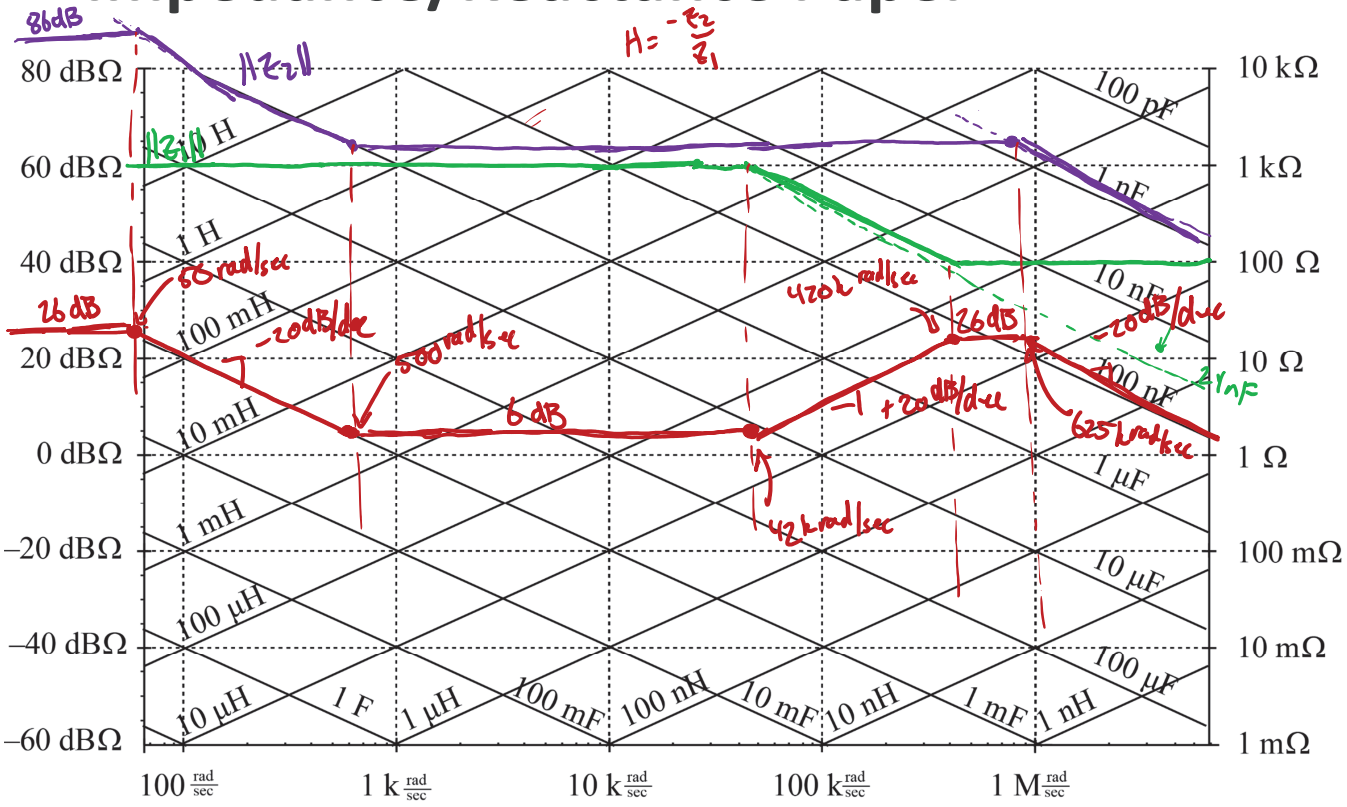




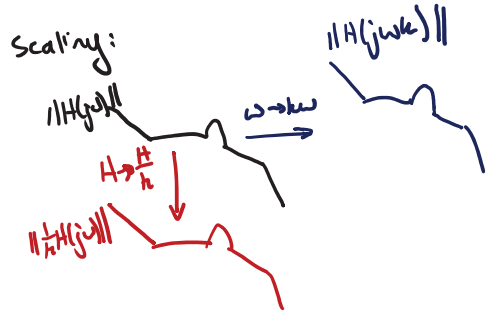


$$H = \frac{V_2(s)}{V_1(s)} = \frac{-Z_2}{Z_1}$$

Impedance/Reactance Paper



Scaling and Bandwidth



frequency Scaling
$\|A(j\omega)\| \to \|H(jk\omega)\|$

$$\begin{cases} R \to R \\ C \to C/k \\ L \to L/k \end{cases}$$

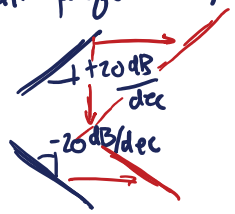
useful to miniaturize

$Z_R = R$ (same for all frequencies)

$Z_L = sL \Rightarrow j\omega L$

$j(\omega/k)L$

$Z_C = \frac{1}{sC} \Rightarrow \frac{-j}{\omega C}$



Magnitude Scaling
$\|H(j\omega)\| \to \frac{1}{k}\|H(j\omega)\|$

$$\begin{cases} R \to R'/k \\ C \to Ck \\ L \to L'/k \end{cases}$$

Bandwidth = $|\omega_2 - \omega_1|$ → some range of frequencies

commonly "-3dB bandwidth" or "half-power bandwidth"

For 2nd order resonance (complex pair)

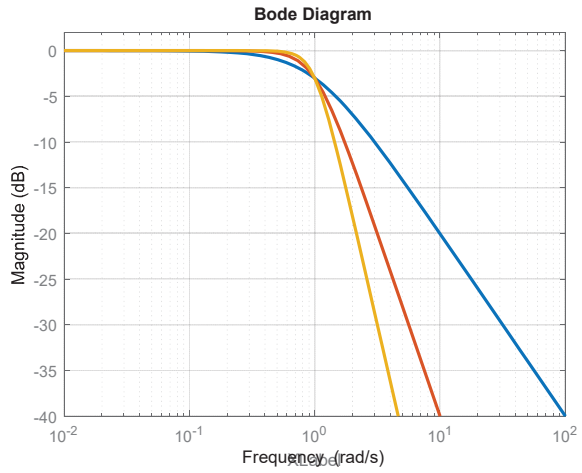
$B = BW > \frac{\omega_0}{Q}$

Higher Q = Less ζ = More frequency selective

Canonical N^{th} Order Filter Designs

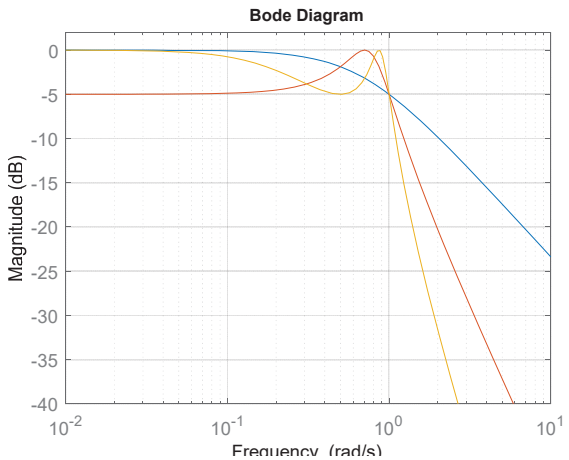
Butterworth

$$|\mathbf{H}(j\omega)| = \frac{K}{\sqrt{1+(\omega/\omega_c)^{2n}}} \qquad n = 1, 2, 3, \ldots$$

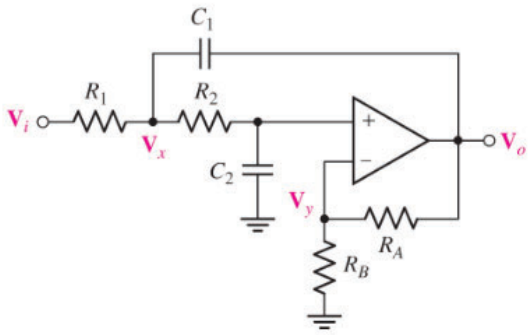


Chebyshev

$$|\mathbf{H}(j\omega)| = \frac{K}{\sqrt{1+\beta^2 C_n^2(\omega/\omega_c)}} \qquad n = 1, 2, 3, \ldots$$



The Sallen-Key Amplifier



$$G \equiv \frac{R_A + R_B}{R_B}$$

$$\frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{G/R_1R_2C_1C_2}{\mathbf{s}^2 + \left[\frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1-G}{R_2C_2}\right]\mathbf{s} + \frac{1}{R_1R_2C_1C_2}}$$

Thank you

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