Plotting a Single Pole Response

R = 10kΩ
C = 10nF

Graphical Construction of Bode Plots

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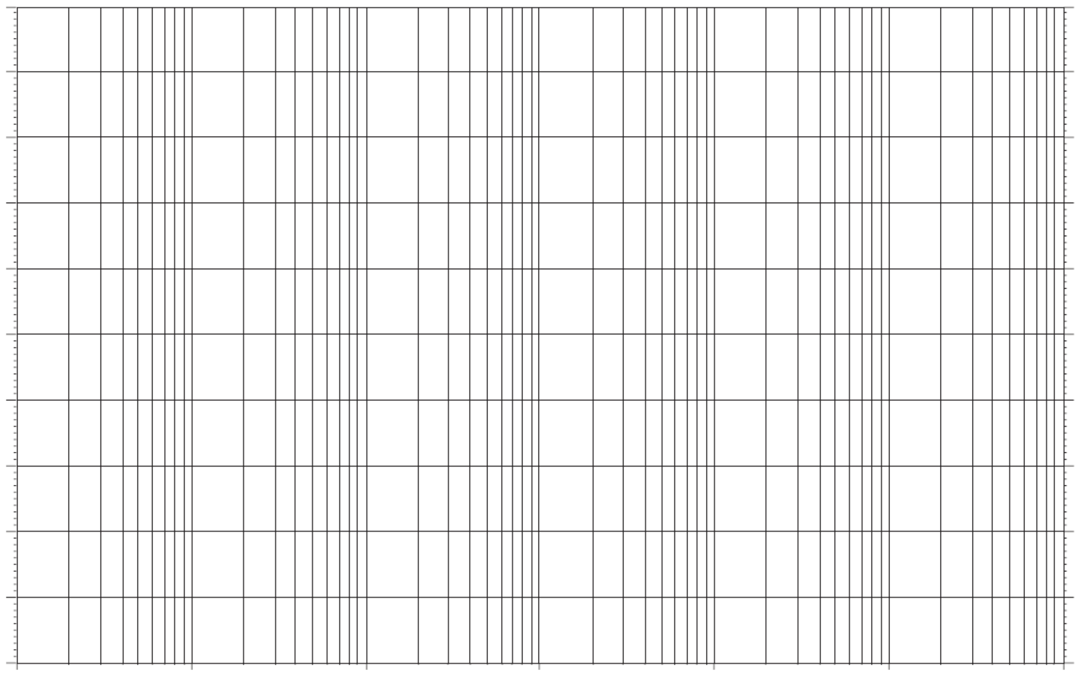
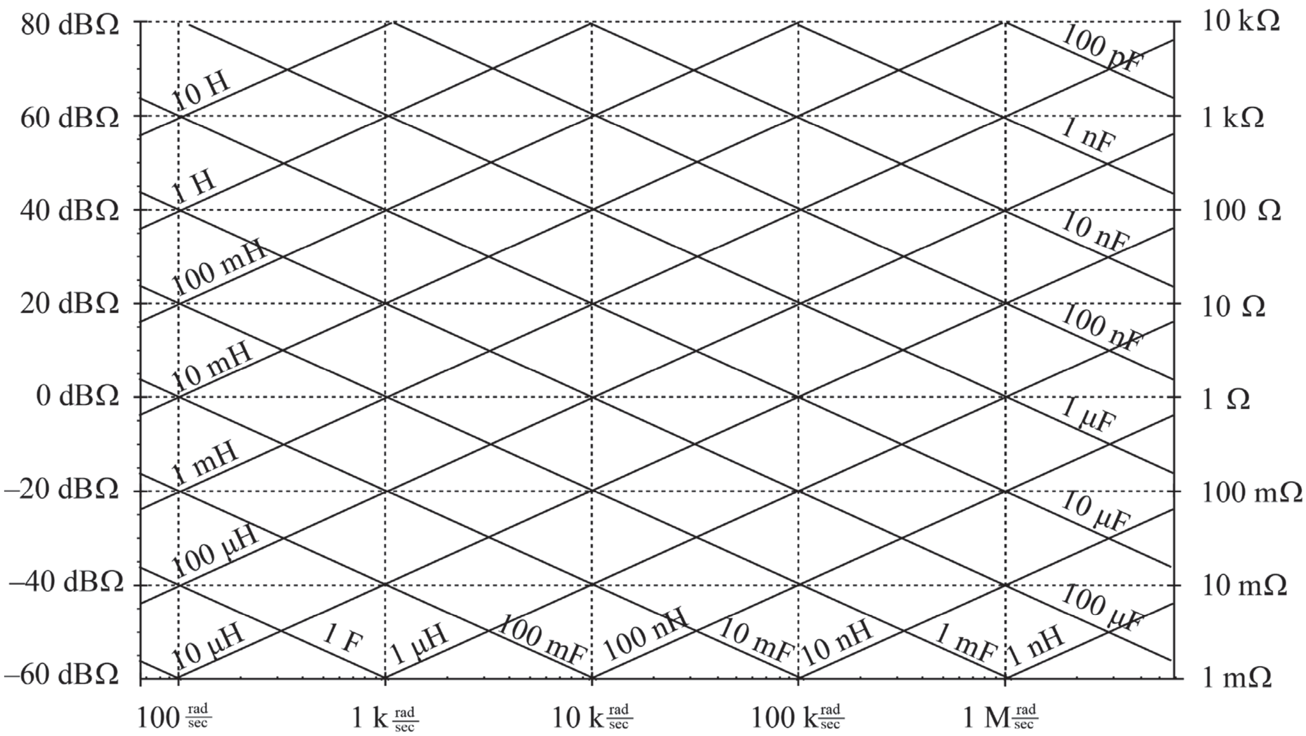
Log Paper



Reactance Paper



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8.1.6 Resonant Poles

Example

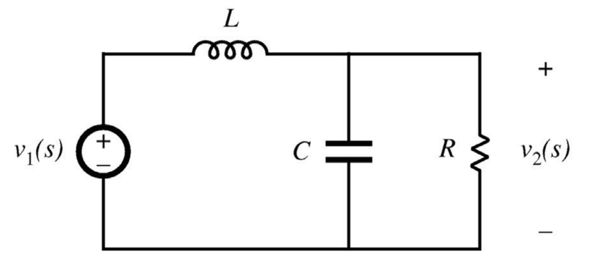
$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

Second-order denominator, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

with $a_1 = L/R$ and $a_2 = LC$

How should we construct the Bode diagram?



Two-pole low-pass filter example

Standard Form for Complex Poles

$$G(s) = \frac{1}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

or

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

- When the coefficients of s are real and positive, then the parameters ζ, ω_0, and Q are also real and positive
- The parameters ζ, ω_0, and Q are found by equating the coefficients of s
- The parameter ω_0 is the angular corner frequency, and we can define $f_0 = \omega_0/2\pi$
- The parameter ζ is called the *damping factor.* ζ controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $\zeta < 1$.
- In the alternative form, the parameter Q is called the *quality factor.* Q also controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $Q > 0.5$.

The Q Factor

In a second-order system, ζ and Q are related according to

$$Q = \frac{1}{2\zeta}$$

Q is a measure of the dissipation in the system. A more general definition of Q, for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that Q has a simple interpretation in the Bode diagrams of second-order transfer functions.

Magnitude Asymptotes

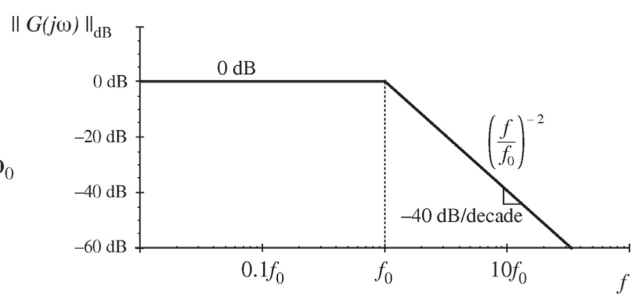
In the form $G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$

let $s = j\omega$ and find magnitude: $\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$

Asymptotes are

$\left\| G \right\| \to 1 \quad \text{for } \omega \ll \omega_0$

$\left\| G \right\| \to \left(\frac{f}{f_0}\right)^{-2} \quad \text{for } \omega \gg \omega_0$



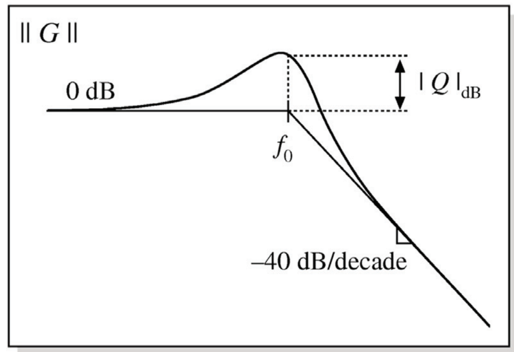
Exact Magnitude Curve

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

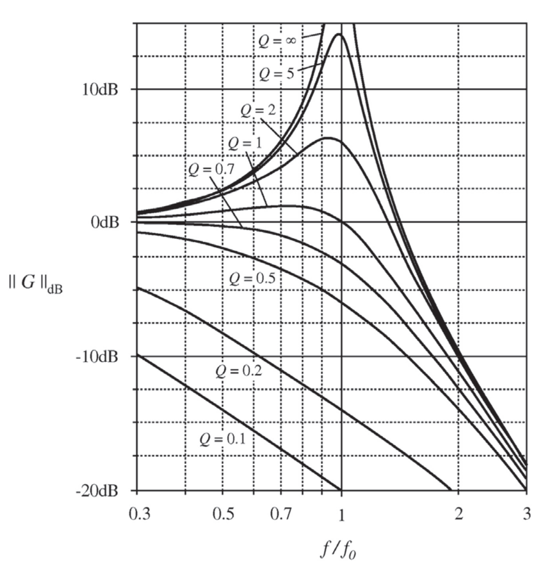
At $\omega = \omega_0$, the exact magnitude is

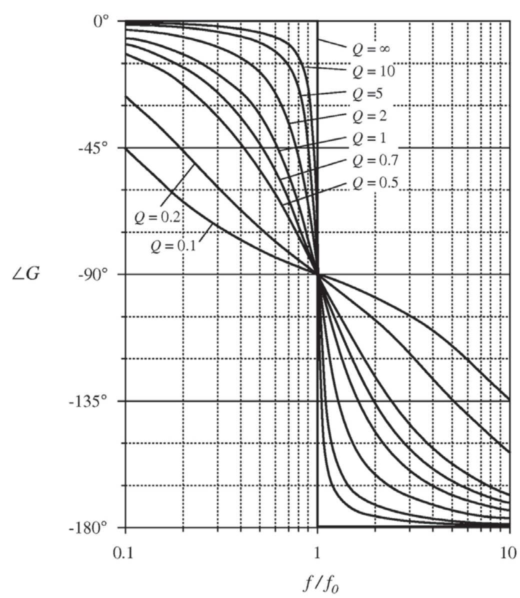
$$\left\| G(j\omega_0) \right\| = Q \qquad \text{or, in dB:} \qquad \left\| G(j\omega_0) \right\|_{dB} = \left| Q \right|_{dB}$$

The exact curve has magnitude Q at $f = f_0$. The deviation of the exact curve from the asymptotes is $| Q |_{dB}$



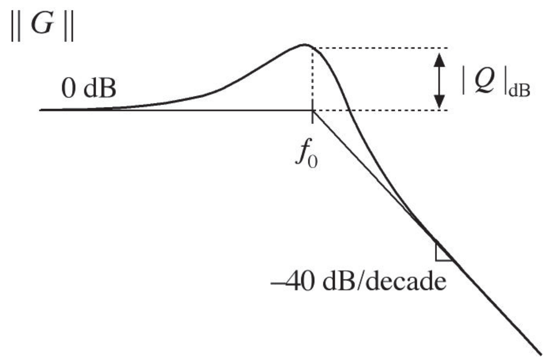
Curves for Varying Q



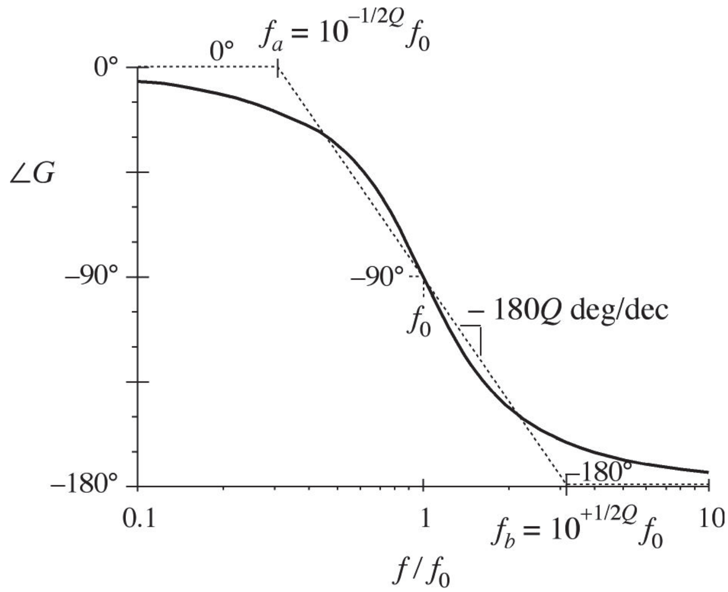


Asymptotes for Complex Poles, Q>0.5

Magnitude



Phase



The Low Q Approximation

Given a second-order denominator polynomial, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

When the roots are real, i.e., when $Q < 0.5$, then we can factor the denominator, and construct the Bode diagram using the asymptotes for real poles. We would then use the following normalized form:

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

This is a particularly desirable approach when $Q \ll 0.5$, i.e., when the corner frequencies ω_1 and ω_2 are well separated.

Derivation of Low-Q Approximation

Given

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Use quadratic formula to express corner frequencies ω_1 and ω_2 in terms of Q and ω_0 as:

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2} \qquad \omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

Corner Frequency ω_1

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

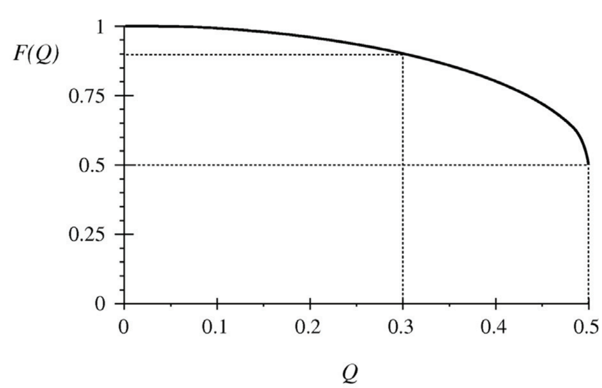
$$\omega_1 = \frac{Q\,\omega_0}{F(Q)}$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

For small Q, $F(Q)$ tends to 1. We then obtain

$$\omega_1 \approx Q\,\omega_0 \qquad \text{for } Q << \frac{1}{2}$$



For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.

Corner Frequency ω_2

$$\omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

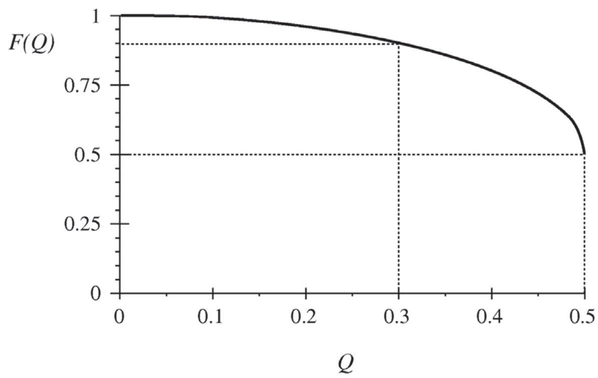
$$\omega_2 = \frac{\omega_0}{Q} F(Q)$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

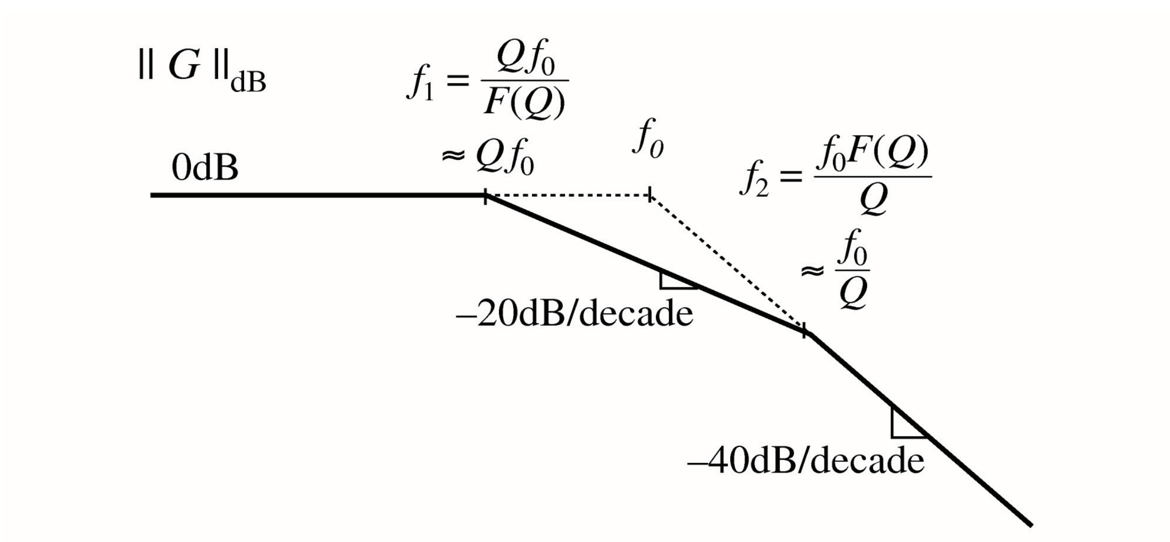
For small Q, $F(Q)$ tends to 1.
We then obtain

$$\omega_2 \approx \frac{\omega_0}{Q} \quad \text{for} \quad Q << \frac{1}{2}$$



For $Q < 0.3$, the approximation $F(Q) = 1$ is within 10% of the exact value.

The Low-Q Approximation



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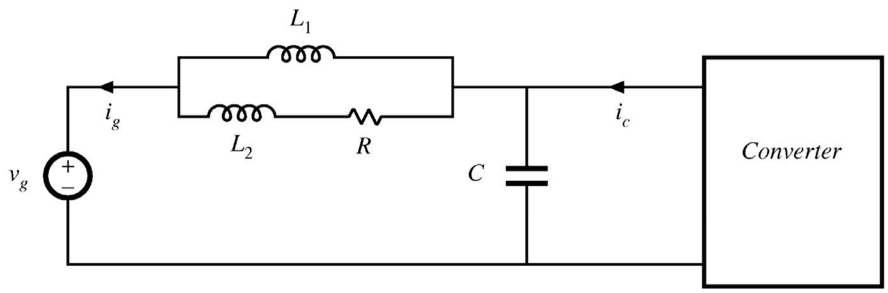
Example: Damped Input EMI Filter



$$G(s) = \frac{i_g(s)}{i_c(s)} = \frac{1 + s\dfrac{L_1 + L_2}{R}}{1 + s\dfrac{L_1 + L_2}{R} + s^2 L_1 C + s^3 \dfrac{L_1 L_2 C}{R}}$$

8.1.8: Approximate Roots of a Polynomial

Generalize the low-Q approximation to obtain approximate factorization of the n^{th}-order polynomial

$$P(s) = 1 + a_1 s + a_2 s^2 + \cdots + a_n s^n$$

It is desired to factor this polynomial in the form

$$P(s) = \left(1 + \tau_1 s\right)\left(1 + \tau_2 s\right) \cdots \left(1 + \tau_n s\right)$$

When the roots are real and well separated in value, then approximate analytical expressions for the time constants τ_1, τ_2, ... τ_n can be found, that typically are simple functions of the circuit element values.

Objective: find a general method for deriving such expressions. Include the case of complex root pairs.

Derivation of the Approximation

Multiply out factored form of polynomial, then equate to original form (equate like powers of s):

$$a_1 = \tau_1 + \tau_2 + \cdots + \tau_n$$
$$a_2 = \tau_1\left(\tau_2 + \cdots + \tau_n\right) + \tau_2\left(\tau_3 + \cdots + \tau_n\right) + \cdots$$
$$a_3 = \tau_1\tau_2\left(\tau_3 + \cdots + \tau_n\right) + \tau_2\tau_3\left(\tau_4 + \cdots + \tau_n\right) + \cdots$$
$$\vdots$$
$$a_n = \tau_1\tau_2\tau_3\cdots\tau_n$$

- Exact system of equations relating roots to original coefficients
- Exact general solution is hopeless
- Under what conditions can solution for time constants be easily approximated?

Case When All Roots Separate

System of equations:

(from previous slide)

$$a_1 = \tau_1 + \tau_2 + \cdots + \tau_n$$
$$a_2 = \tau_1\left(\tau_2 + \cdots + \tau_n\right) + \tau_2\left(\tau_3 + \cdots + \tau_n\right) + \cdots$$
$$a_3 = \tau_1\tau_2\left(\tau_3 + \cdots + \tau_n\right) + \tau_2\tau_3\left(\tau_4 + \cdots + \tau_n\right) + \cdots$$
$$\vdots$$
$$a_n = \tau_1\tau_2\tau_3\cdots\tau_n$$

Suppose that roots are real and well-separated, and are arranged in decreasing order of magnitude:

$$\left|\tau_1\right| >> \left|\tau_2\right| >> \cdots >> \left|\tau_n\right|$$

Then the first term of each equation is dominant

$\Rightarrow$ Neglect second and following terms in each equation above

Approximation When Roots are Well Separated

System of equations:

(only first term in each equation is included)

$$a_1 \approx \tau_1$$
$$a_2 \approx \tau_1 \tau_2$$
$$a_3 \approx \tau_1 \tau_2 \tau_3$$
$$\vdots$$
$$a_n = \tau_1 \tau_2 \tau_3 \cdots \tau_n$$

Solve for the time constants:

$$\tau_1 \approx a_1$$
$$\tau_2 \approx \frac{a_2}{a_1}$$
$$\tau_3 \approx \frac{a_3}{a_2}$$
$$\vdots$$
$$\tau_n \approx \frac{a_n}{a_{n-1}}$$

Results

If the following inequalities are satisfied

$$\left| a_1 \right| >> \left| \frac{a_2}{a_1} \right| >> \left| \frac{a_3}{a_2} \right| >> \cdots >> \left| \frac{a_n}{a_{n-1}} \right|$$

Then the polynomial $P(s)$ has the following approximate factorization

$$P(s) \approx \left(1 + a_1 s\right) \left(1 + \frac{a_2}{a_1} s\right) \left(1 + \frac{a_3}{a_2} s\right) \cdots \left(1 + \frac{a_n}{a_{n-1}} s\right)$$

- If the a_n coefficients are simple analytical functions of the element values L, C, etc., then the roots are similar simple analytical functions of L, C, etc.
- Numerical values are used to justify the approximation, but analytical expressions for the roots are obtained

Quadratic Roots: Not Well Separated

Suppose inequality k is not satisfied:

$$\left|a_1\right| >> \left|\frac{a_2}{a_1}\right| >> \cdots >> \left|\frac{a_k}{a_{k-1}}\right| \not>> \left|\frac{a_{k+1}}{a_k}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

↑ not satisfied

Then leave the terms corresponding to roots k and $(k+1)$ in quadratic form, as follows:

$$P(s) \approx \left(1 + a_1 s\right)\left(1 + \frac{a_2}{a_1} s\right) \cdots \left(1 + \frac{a_k}{a_{k-1}} s + \frac{a_{k+1}}{a_{k-1}} s^2\right) \cdots \left(1 + \frac{a_n}{a_{n-1}} s\right)$$

This approximation is accurate provided

$$\left|a_1\right| >> \left|\frac{a_2}{a_1}\right| >> \cdots >> \left|\frac{a_k}{a_{k-1}}\right| >> \left|\frac{a_{k-2}\,a_{k+1}}{a_{k-1}^2}\right| >> \left|\frac{a_{k+2}}{a_{k+1}}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

First Inequality Violated

When inequality 1 is not satisfied:

$$\left|a_1\right| \not>> \left|\frac{a_2}{a_1}\right| >> \left|\frac{a_3}{a_2}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

↑ not satisfied

Then leave the first two roots in quadratic form, as follows:

$$P(s) \approx \left(1 + a_1 s + a_2 s^2\right)\left(1 + \frac{a_3}{a_2} s\right) \cdots \left(1 + \frac{a_n}{a_{n-1}} s\right)$$

This approximation is justified provided

$$\left|\frac{a_2^2}{a_3}\right| >> \left|a_1\right| >> \left|\frac{a_3}{a_2}\right| >> \left|\frac{a_4}{a_3}\right| >> \cdots >> \left|\frac{a_n}{a_{n-1}}\right|$$

Other Cases

- Several nonadjacent inequalities violated
  - Apply same process multiple times
- Multiple adjacent inequalities violated
  - More than two roots close in value
  - Must use 3rd order or higher polynomial