Neglecting the switching ripple

Suppose the duty cycle is modulated sinusoidally:

\[ d(t) = D + D_m \cos \omega_m t \]

where \( D \) and \( D_m \) are constants, \( |D_m| \ll D \), and the modulation frequency \( \omega_m \) is much smaller than the converter switching frequency \( \omega_s = 2\pi f_s \).

The resulting variations in transistor gate drive signal and converter output voltage:

\[ \text{Gate drive} \]

\[ \text{Actual waveform } v(t) \text{ including ripple} \]

\[ \text{Averaged waveform } (v(t))_a \text{ with ripple neglected} \]

---

Output voltage spectrum with sinusoidal modulation of duty cycle

Contains frequency components at:
- Modulation frequency and its harmonics
- Switching frequency and its harmonics
- Sidebands of switching frequency

With small switching ripple, high-frequency components (switching harmonics and sidebands) are small. If ripple is neglected, then only low-frequency components (modulation frequency and harmonics) remain.
Sampling

\[ v^*(t) = v(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

Unit impulse (Dirac)

Sampled-data system example: frequency domain

\[ v^*(s) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} v(s - jk\omega_s) \]

\[ v_0(s) = \frac{1-e^{-st}}{s} \]

Consider only low-frequency signals: \[ v_0(s) \approx \frac{1-e^{-st}}{st} v(s) \]

System “transfer function” \[ \frac{v_0}{v} = \frac{1-e^{-st}}{st} \]
Zero-order hold: frequency responses

\[ f_s = 1 \text{ MHz} \]

Zero-Order Hold magnitude and phase responses

\[ H / T = \frac{1 - e^{-sT}}{sT} \]

How does any of this apply to converter modeling?
PWM is a small-signal sampler!

PWM sampling occurs at \( t_p \) (i.e. at \( dT_s \), periodically, in each switching period)

General sampled-data model

- Sampled-data model valid at all frequencies
- **Equivalent hold** describes the converter small-signal response to the sampled duty-cycle perturbations [Billy Lau, PESC 1986]
- State-space averaging or averaged-switch models are low-frequency continuous-time approximations to this sampled-data model
Application to DCM high-frequency modeling

\[ i_L \]

\[ c \]

\[ dT_s \quad d_2T_s \quad \frac{T_s}{T_s} \]
DCM inductor current high-frequency response

\[ \dot{i}_{L}(s) = \frac{V_{1} + V_{2}}{L} T_{i} \frac{1 - e^{-sT_{i}}}{s} \hat{d}(s) = \frac{V_{1} + V_{2}}{L} T_{i} \frac{1 - e^{-sT_{i}}}{s} \sum_{k=1}^{\infty} \hat{d}(s - jk\omega_{2}) \]

\[ \dot{i}_{L}(s) \approx \frac{V_{1} + V_{2}}{L} D_{1} T_{i} \frac{1 - e^{-sT_{i}}}{s} \hat{d}(s) \]

\[ \frac{\dot{i}_{L}(s)}{\hat{d}(s)} \approx \frac{V_{1} + V_{2}}{L} D_{1} T_{i} \frac{1}{1 + \frac{s}{\omega_{2}}} \quad \omega_{2} = \frac{2}{D_{1} T_{i}} \]

\[ f_{2} = \frac{f_{s}}{\pi D_{2}} \]

High-frequency pole due to the inductor current dynamics in DCM, see (11.77) in Section 11.3

Example DAB Application

- Goal: Include effects of both output sampling and PWM sampling in converter model
- Considering only control perturbations, model will take the form

\[ \ddot{x}[n] = F\ddot{x}[n-1] + G\dot{\phi}_{ab}[n-1] \]
Example DAB Application

- Goal is to develop a model of the form

\[ \hat{x}[n] = F\hat{x}[n-1] + G\hat{\phi}_{ab}[n-1] \]

- State vector is sampled and control updated at \( t = nT_s \)

\[ x = \begin{bmatrix} v_p & i_i & v_{out} \end{bmatrix}^T \]

Continuous Time State Space

\[ \dot{x}(t) = A_t x(t) + B_t V_g \]
Continuous Time State Representation

\[ \dot{x}(t) = A(t)x(t) + B(t)V_g \]

- Models change slightly depending on operating mode
- In all cases, consider dynamics at boundary just below ZVS

State Propagation

- Within each subinterval, states propagate according to continuous time model
  \[ \dot{x}(t) = A(t)x(t) + B(t)V_g \]
- Solution given by
  \[ x(t) = e^{A(t)T}x_0 + \int_0^T e^{-A(t-T)} \left( B(t)V_g(T) \right) dT. \]
- If \( V_g(t) \approx V_g \) within one switching period,
  \[ x_1(t_1) = e^{A(t_1)}x_0 + A(t_1)^{-1}\left( e^{A(t_1)} - I \right)B(t_1)V_g \]

Subinterval:
- \( \sigma_1 \)
- \( \sigma_2 \)
- \( \sigma_3 \)
- \( \sigma_4 \)
- \( \sigma_5 \)
- \( \sigma_6 \)
Small Signal State Propagation

- Considering steady state plus some small signal, solution becomes
  \[ X_1 + \hat{x}_0 = e^{A t} (X_0 + \hat{x}_0) + A_1^{-1} (e^{A_1 t} - I) B_1 V_g \]

- Small signal portion can be separated out
  \[ \hat{x}_1 = e^{A_1 t} \hat{x}_0 \]

- And applied for each subinterval to obtain natural response

\[ F = e^{A t} e^{A_1 t} e^{A_2 t} e^{A_3 t} e^{A_4 t} e^{A_5 t} \]

Small Signal Response to Phase Perturbation

- For the forced response, consider the effect of an increase in phase shift interval of magnitude \( \phi_{ab} \) around steady-state vector \( X_{p1} \)
Small Signal Response to Phase Perturbation

- Perturbation in phase shift $\hat{\phi}_{ab}$ results in a state perturbation $\hat{x}_{d1}$
- In accordance with small signal modeling assumptions, affect on phase perturbations on state assumed to be instantaneous

$$\hat{x}_{d1} = (A_2 - A_3) X_{p1} \frac{T_s}{2\pi} \hat{\phi}_{ab}$$

Subinterval: I II IV V VI VIII

Small Signal Response to Phase Perturbation

- Perturbation in phase shift $\hat{\phi}_{ab}$ results in a state perturbation $\hat{x}_{d1}$
- In accordance with small signal modeling assumptions, affect on phase perturbations on state assumed to be instantaneous
- State perturbation is then similarly propagated through state progression to next sampling interval

$$x_{[n]} \hat{x}_{d1} = e^{4\phi_{ax}} e^{\Delta\phi_{ax}} (A_2 - A_3) X_{p1} \frac{T_s}{2\pi} \hat{\phi}_{ab}$$
Small Signal Response to Phase Perturbation

- Perturbation in phase shift $\phi_{ab}$ results in a state perturbation $\delta x_{d1}$
- In accordance with small signal modeling assumptions, affect on phase perturbations on state assumed to be instantaneous
- Perturbations propagated to output for two independent changes in state at next sampling instant

$$G = e^{A_{d1} T_s} e^{A_{d2} T_s} \left( A_2 - A_3 \right) X_{p1} \frac{T_s}{2\pi} + e^{A_{d3} T_s} \left( A_5 - A_6 \right) X_{p2} \frac{T_s}{2\pi}$$

Complete Discrete Time Model

- Combining forced and natural responses into
- Full model is obtained

$$\hat{x}[n] = F\hat{x}[n-1] + G\hat{\phi}_{ab}[n-1]$$

$$\hat{x}[n] = \left( e^{A_{d1} T_s} e^{A_{d2} T_s} e^{A_{d3} T_s} e^{A_{d4} T_s} \right) \hat{x}[n-1] + \cdots$$

$$\left( e^{A_{d1} T_s} e^{A_{d2} T_s} \left( A_2 - A_3 \right) X_{p1} \frac{T_s}{2\pi} + e^{A_{d3} T_s} \left( A_5 - A_6 \right) X_{p2} \frac{T_s}{2\pi} \right) \hat{\phi}_{ab}$$

- Model can be further simplified through recognition of half-cycle symmetry
Half-Cycle Symmetry

- All states are either symmetric or antisymmetric about the half-period
- Can sample and control at twice the switching frequency:

\[ x\left(t + \frac{T_s}{2}\right) = I_{HC}x(t) \]

\[ I_{HC} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \hat{x}[n] = \left(e^{A_1} - e^{A_2}I_{HC}\right)\hat{x}[n-1] + \left(e^{A_1} - A_3\right)x_{p1}\frac{T_s}{2\pi}\hat{\phi}_{ab}[n-1] \]

DAB: Experimental Results


**Frequency Response Comparison**

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<tr>
<th>$P_{out} = 30\ W$</th>
<th>$P_{out} = 90\ W$</th>
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</thead>
</table>

- Bode plot compared to traditional averaged model (neglects ZVS dynamics)
- Experimental results verify very little change in dynamics other than DC gain

**Transient Response Comparison**

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<tr>
<th>$P_{out} = 30\ W$</th>
<th>$P_{out} = 90\ W$</th>
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- Response in current and voltage shown at 30 W (left) and 90 W (right) for step changes in $\phi_{ab}$
- Both well matched to experimental results
Additional Readings


Power Electronics Courses at UTK

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<th>Junior</th>
<th>Senior</th>
<th>Graduate</th>
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<td>ECE 481 Power Electronics</td>
<td>ECE 523 Power Electronics and Drives</td>
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<tr>
<td>ECE 482 / 599 Power Electronic Circuits</td>
<td>ECE 525 Alternative Energy Sources</td>
<td>ECE 623 Advanced Power Electronics and Drives</td>
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