Discrete Time Modeling of a Buck Converter with Large Ripple

Fig. 1 shows a buck converter with inductor and capacitor series resistance $R_L$ and $R_{esr}$, respectively. The converter operates with $V_g=10\text{V}$, $D=0.2$, $f_s=1\text{MHz}$, $R_{esr}=16\text{m\Omega}$, $R_L=25\text{m\Omega}$, $L=1\text{\mu H}$, and $C_{out}=50\text{nF}$. The duty cycle of $Q_1$ is $D$. $V_g$ is DC-only (i.e. has zero small-signal component). $Q_1$ and $Q_4$ are ideal devices. The output resistance is $R=1\Omega$

For parts (a) and (b), you may use typical (ECE 481) analysis techniques, including volt-second balance, cap-charge balance, and small-ripple approximation.

a) Derive expressions and compute values for the stead-state ripples of $i_l$ and $v_c$. You may neglect resistance $R_L$ and $R_{esr}$ for this step.

b) Derive the averaged, small-signal control-to-output transfer function of the buck converter, $G_{vd}(s)$

For parts (c)-(e), you may not use averaging or small ripple approximations.

c) Using the state vector $x = [v_c \ i_l]^T$, and using the input source as the only independent input ($u = V_g$) derive matrices $A_i$ and $B_i$ for each switching subinterval which model the subinterval behavior in the form

$$\dot{x} = Ax + Bu$$

State the range of times over which each state description is valid within one period, $0 < t < T_s$

d) Solve for a matrix $C$ such that $v_{out} = Cx$

e) Write a closed-form expression, in terms of the matrices derived in (c), $V_g$, $D$, and $T_s$, for the exact steady-state state vector $x_0=x(0)$ at the start of the switching period. Expand all product and summations ($[\prod, \Sigma$), and eliminate any zero terms. Using MATLAB or any other tool, solve the exact value of $v_c$ and $i_l$ at the start of the switching period.

f) Write a closed-form expression, in terms of the matrices derived in (c), $X_0$, $V_g$, $D$, and $T_s$, for the exact steady-state state vector $x(DT_s)$ at the switching instant. Using MATLAB or any other tool, solve the exact value of $v_c$ and $i_l$ at the start of the switching period.

g) Using the discrete-time method discussed in class, write expressions for matrices $\Phi$ and $\gamma$ that complete the small-signal, discrete time model

$$\hat{x}[n] = \Phi\hat{x}[n-1] + \gamma d[n-1]$$

h) Using Matlab or any other tool, create a Bode plot on the same axes of $G_{va}(s)$ derived in (b) and $G_{va}(z) = C(zI-\Phi)^{-1}\gamma$.

i) Comment on the differences between the transfer functions in continuous time and discrete time, and the differences in calculated ripple in (a) and (e-f).
Useful MATLAB functions:

<table>
<thead>
<tr>
<th>MATLAB Function</th>
<th>Description</th>
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<tbody>
<tr>
<td>expm(X)</td>
<td>matrix exponential of X</td>
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<tr>
<td>ss(A,B,C,D,Ts)</td>
<td>creates a discrete-time state-space model with sample time Ts</td>
</tr>
<tr>
<td>tf('s')</td>
<td>specifies the transfer function H(s) = s (Laplace variable)</td>
</tr>
<tr>
<td>tf('z',Ts)</td>
<td>specifies H(z) = z with sample time Ts</td>
</tr>
<tr>
<td>eye(N)</td>
<td>N-by-N identity matrix</td>
</tr>
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Example code generating the bode plot of a capacitor impedance:

```matlab
Cout = 50e-9;
s = tf('s');
Zc = 1/(s*Cout);
bode(Zc);
```

Example code generating the bode plot of a discrete-time system defined by \( \Phi \), \( \gamma \), and \( C \), with sampling rate \( T_s \):

```matlab
G = ss(PHI, GAMMA, C, 0, Ts);
bode(G);
```