Historical Perspective

Robert D Middlebrook
PhD, Standford, 1955
CalTech Professor, 1955-1998

Slobodan Cúk
PhD CalTech, 1976
CalTech Prof, 1977-1999

Modelling, analysis, and design of switching converters

Model a switched system as an averaged, time-invariant system with

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

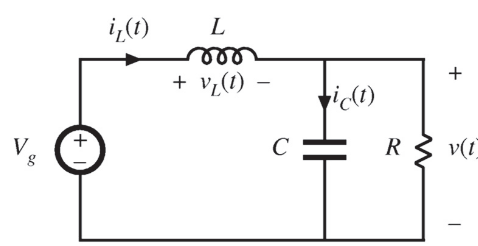
where

$$\boldsymbol{A} = D\boldsymbol{A}_1 + D'\boldsymbol{A}_2$$
$$\boldsymbol{B} = D\boldsymbol{B}_1 + D'\boldsymbol{B}_2$$

Linear Circuit Modeling Using State Space

In switch position 1

$$\begin{cases} v_g(t) - v_c(t) = L\dfrac{di_L(t)}{dt} \\ i_L(t) - \dfrac{v_c(t)}{R} = C\dfrac{dv_c(t)}{dt} \end{cases}$$



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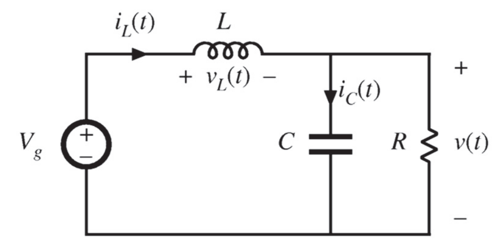


Which can be written, in state space, form as

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1}{L} \\ \dfrac{1}{C} & \dfrac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v_g(t)$$

Or, generally,

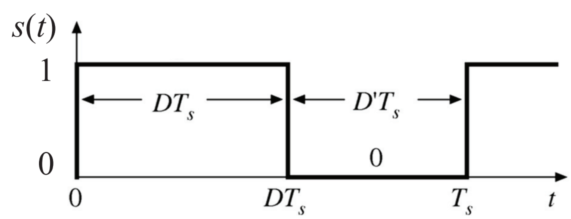
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1 u(t)$$

In the second switch position, we will have a new (linear) circuit with

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2 u(t)$$

Switching Signal

In a PWM converter with two switch positions, the two linear circuits combine according to a switching function $s(t)$



$$\dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_1 s(t) + \boldsymbol{A}_2 s'(t)]\boldsymbol{x}(t) + [\boldsymbol{B}_1 s(t) + \boldsymbol{B}_2 s'(t)]u(t)$$

where

$$s(t) = \begin{cases} 1, & \text{if } \ nT_s < t < (n+D)T_s \\ 0, & \text{if } \ (n+D)T_s < t < (n+1)T_s \end{cases}$$

$$s'(t) = 1 - s(t)$$

SMPS State Space

In traditional state space modeling of linear systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

with $u(t)$ containing a control input. When $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant, this is a linear system. However, we have

$$\dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_1 s(t) + \boldsymbol{A}_2 s'(t)]\boldsymbol{x}(t) + [\boldsymbol{B}_1 s(t) + \boldsymbol{B}_2 s'(t)]u(t)$$

or, equivalently

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)u(t)$$

which is nonlinear: how do we deal with it?

Converting to Linear System

Assume that our system model

$$\dot{\boldsymbol{x}}(t) = [\boldsymbol{A}_1 s(t) + \boldsymbol{A}_2 s'(t)]\boldsymbol{x}(t) + [\boldsymbol{B}_1 s(t) + \boldsymbol{B}_2 s'(t)]u(t)$$

can be approximated by some linear system

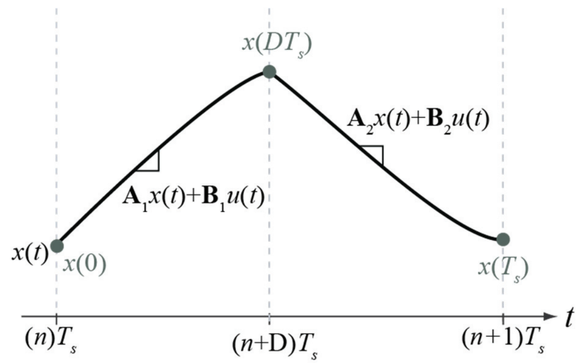
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

which removes the nonlinearity of the system

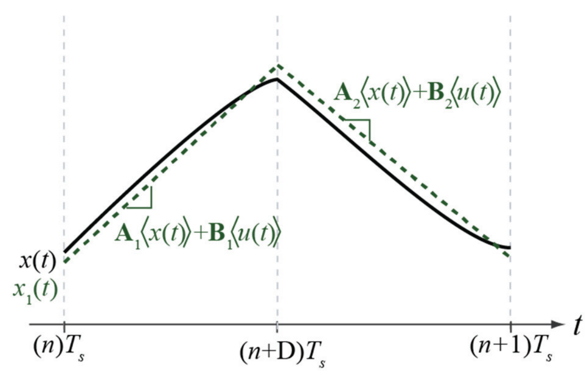
- Nonlinearities came from switching
- Expect that switching dynamics will be lost

Note: This system is now linear in $\boldsymbol{x}(t)$ and $u(t)$, but not in our control signal, $s(t)$

Approximate Steady State Waveforms

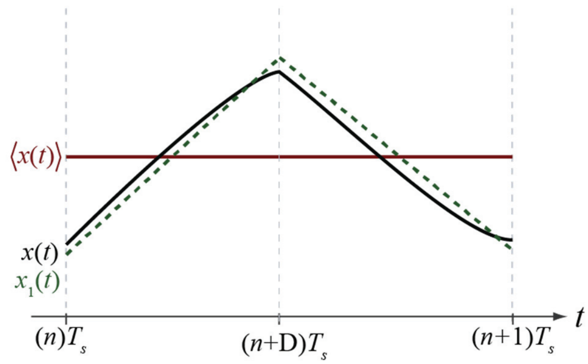


Approximate Steady State Waveforms



$$\langle x(t)\rangle = \frac{1}{T_s}\int_0^{T_s} x(t)dt$$

Approximate Steady State Waveforms



The Averaging Approximation

If waveforms can be approximated as linear

$$\dot{\boldsymbol{x}}(t) = \begin{cases} \boldsymbol{A_1}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B_1}\langle u(t)\rangle, & \text{if } \ nT_s < t < (n+D)T_s \\ \boldsymbol{A_2}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B_2}\langle u(t)\rangle,, & \text{if } \ (n+D)T_s < t < (n+1)T_s \end{cases}$$

so the average slope is

$$\langle \dot{\boldsymbol{x}}(t)\rangle$$

$$= \frac{1}{T_s}(\boldsymbol{A_1}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B_1}\langle u(t)\rangle)DT_s + (\boldsymbol{A_2}\langle \boldsymbol{x}(t)\rangle + \boldsymbol{B_2}\langle u(t)\rangle)(1-D)T_s$$

or, rearranging

$$\langle \dot{\boldsymbol{x}}(t)\rangle = (D\boldsymbol{A_1} + D'\boldsymbol{A_2})\langle \boldsymbol{x}(t)\rangle + (D\boldsymbol{B_1} + D'\boldsymbol{B_2})\langle u(t)\rangle$$

The Averaged System

This equation is now the model of a new, equivalent linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

where

$$\boldsymbol{A} = D\boldsymbol{A}_1 + D'\boldsymbol{A}_2$$
$$\boldsymbol{B} = D\boldsymbol{B}_1 + D'\boldsymbol{B}_2$$

which has averaged behavior over one switching period

This approximation is *perhaps* valid, if

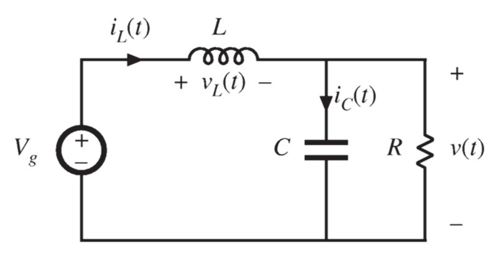
- State waveforms are dominantly linear
- Dynamics of interest are at $f_{bw} \ll f_s$

Buck State Space Averaging

In switch position 1

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{B}_1u(t)$$

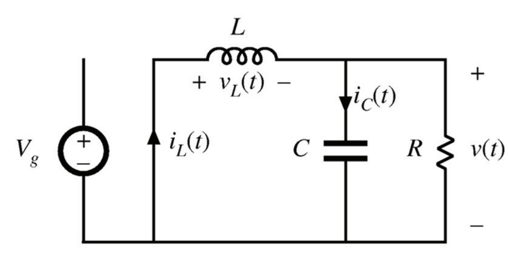
$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v_g(t)$$



In switch position 2

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2u(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_g(t)$$



Buck Averaged Model

So, our average model is

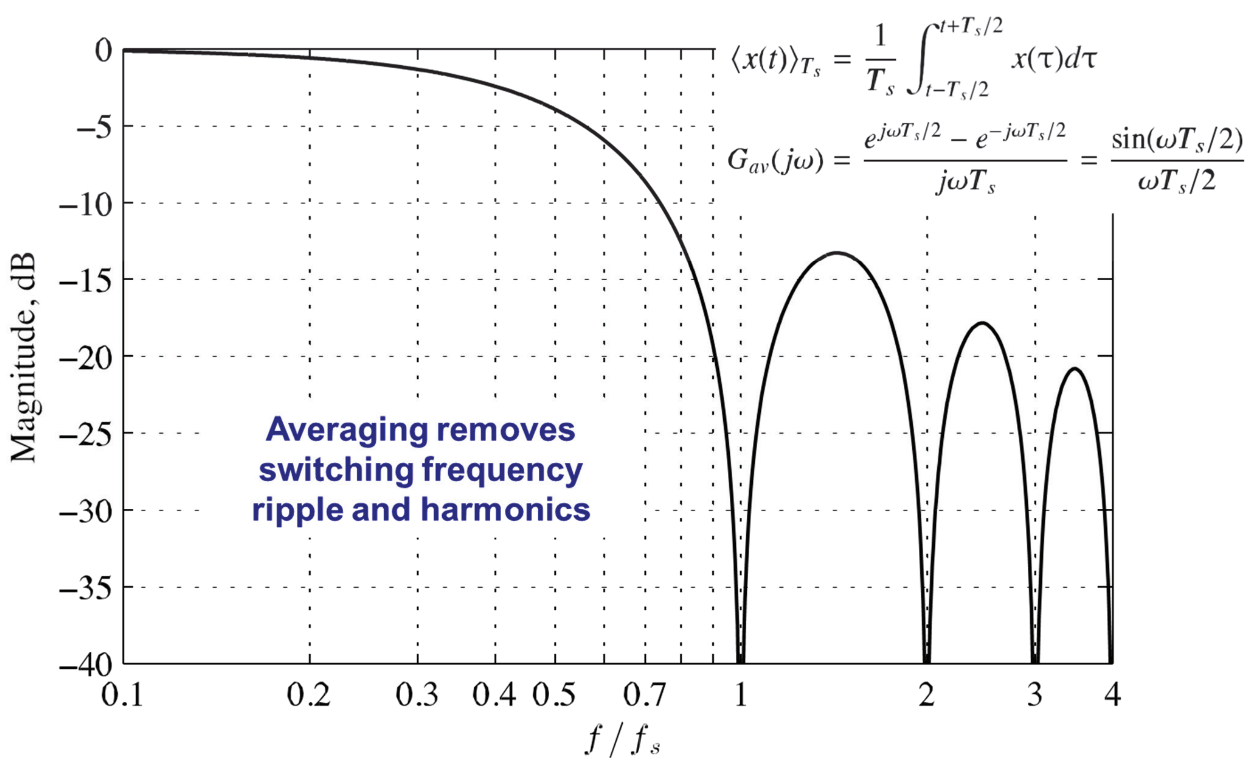
$$\langle \dot{\boldsymbol{x}}(t)\rangle = (D\boldsymbol{A_1} + D'\boldsymbol{A_2})\langle \boldsymbol{x}(t)\rangle + (D\boldsymbol{B_1} + D'\boldsymbol{B_2})\langle u(t)\rangle$$

$$\langle \dot{\boldsymbol{x}}(t)\rangle = \left(D \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} + D' \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \right) \langle \boldsymbol{x}(t)\rangle + \left(D \begin{bmatrix} 1/L \\ 0 \end{bmatrix} + D' \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) V_g$$

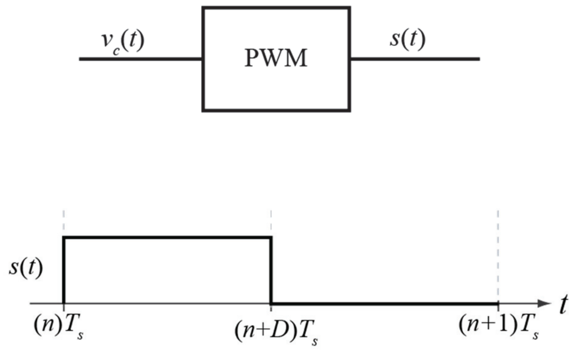
$$\langle \dot{\boldsymbol{x}}(t)\rangle = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \langle \boldsymbol{x}(t)\rangle + \begin{bmatrix} D/L \\ 0 \end{bmatrix} V_g$$

$$\begin{cases} DV_g - \langle v_c(t)\rangle = L\dfrac{d\langle i_L(t)\rangle}{dt} \\ \langle i_L(t)\rangle - \dfrac{\langle v_c(t)\rangle}{R} = C\dfrac{d\langle v_c(t)\rangle}{dt} \end{cases}$$

Averaging: Discussion



Discrete Time Nature of PWM



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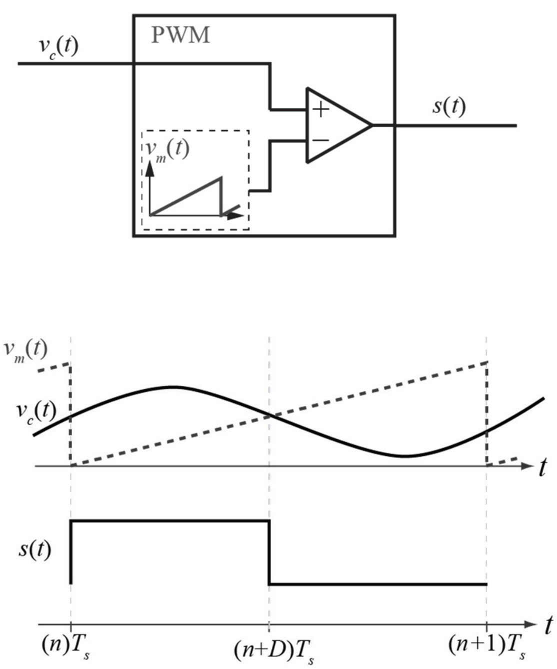
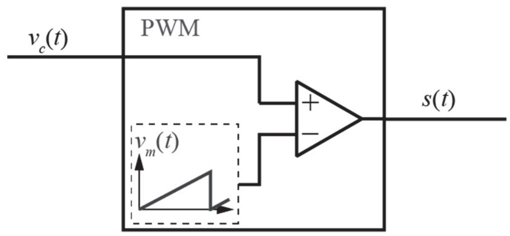
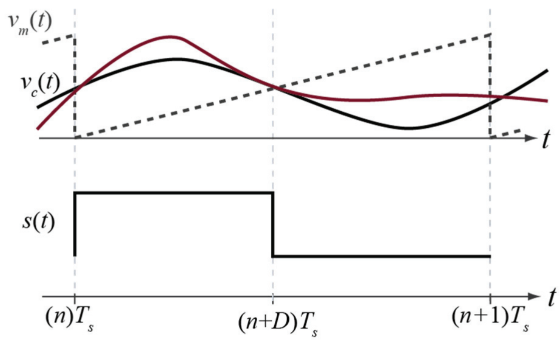


Discrete Time Nature of PWM





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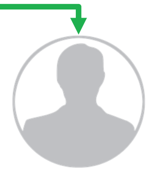


Dennis John Packard
PhD, CalTech 1976

Discrete modeling and analysis of switching regulators

Model a switched system as a discrete-time system with

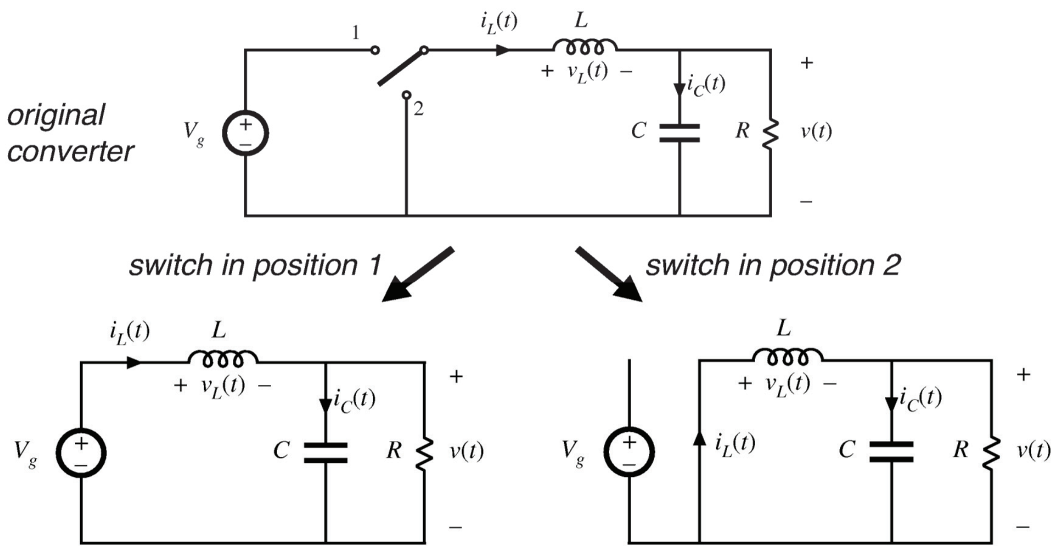
$$\boldsymbol{x}[n+1] = \boldsymbol{\Phi}\boldsymbol{x}[n] + \boldsymbol{\Psi}U[n]$$

where

$$\boldsymbol{\Phi} = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)$$
$$\boldsymbol{\Psi} = \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i^{-1}\left(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I}\right)\boldsymbol{B}_i\right\}$$

A. R. Brown and R. D. Middlebrook, "Sampled-data Modeling of Switching Regulators" PESC 1981

Large Signal Modeling of SMPS



Discrete Time Modeling

- Every subcircuit is a passive, linear circuit
- Passive, linear circuits can be solved in closed-form
  - Can model states at discrete times without averaging
- Only assumptions required
  - Independent inputs are DC or slowly varying

Solution to State Space Equation

Closed form solution to state space equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)$$

Multiply both sides by $e^{-\boldsymbol{A}t}$

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Left-hand side is

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Solution to State Space Equation

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}u(t)$$

Can now be solved by direct integration

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}u(\tau)\,d\tau$$

Rearranging

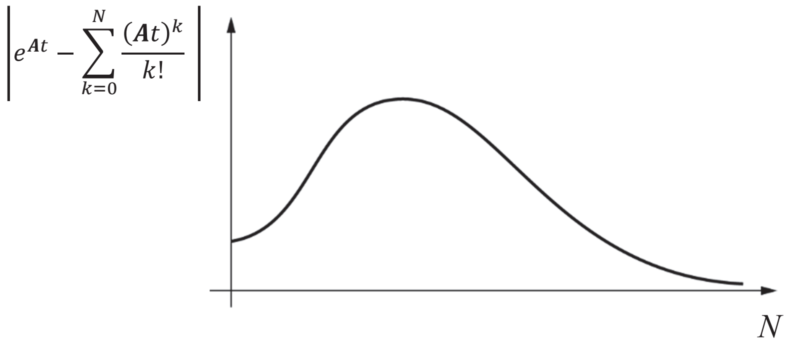
$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{-\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)\,d\tau$$

Matrix Exponential

Matrix exponential defined by Taylor series expansion

$$e^{\mathbf{A}t} = I + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \cdots + \frac{(\mathbf{A}t)^N}{N!} = \sum_{k=0}^{N} \frac{(\mathbf{A}t)^k}{k!}$$

Well-known issue with convergence in many cases



C. Moler and C. V. Loan, “Nineteen dubious ways to compute the exponential of a matrix,” SIAM Review, vol. 20, pp. 801–836, 1978.

Properties of the Matrix Exponential

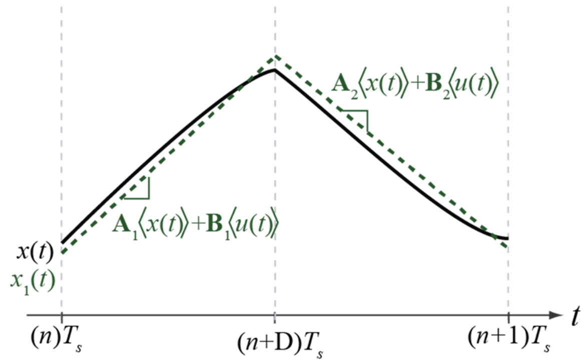
- Matrix exponential always exists
  - i.e. summation will always converge
- Exponential of any matrix is always invertible, with

$$e^{\mathbf{A}} e^{-\mathbf{A}} = \mathbf{I}$$

First Order Taylor Series Expansion

Linear ripple approximation

$$e^{At} \approx I + At$$



Valid only if switching frequency much faster than system modes

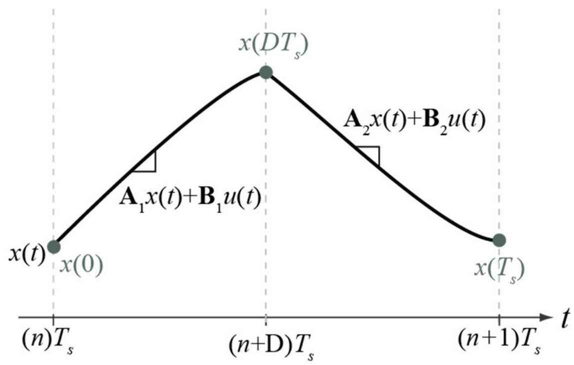
Simplification for Slow-Varying Inputs

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{-A(t-\tau)}\boldsymbol{B}u(\tau)\,d\tau$$

If A is invertible and $u(\tau) \approx U$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \boldsymbol{A}^{-1}(e^{At} - \boldsymbol{I})\boldsymbol{B}U$$

Application to Switching Converter



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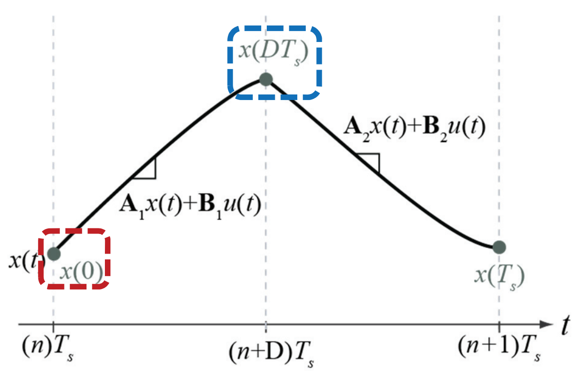

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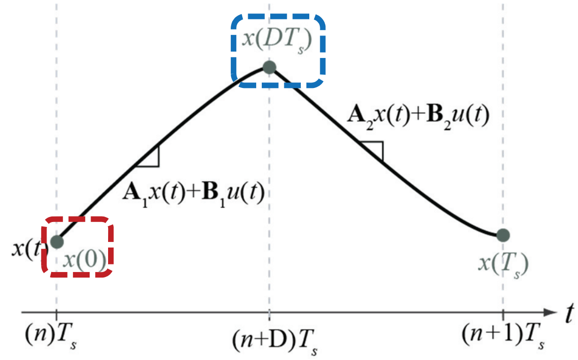
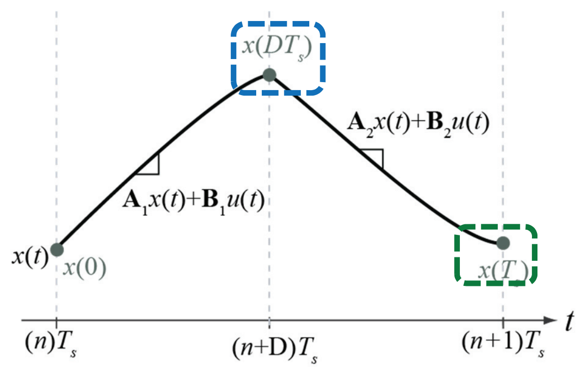


$$x(DT_s) = e^{A_1 DT_s} x(0) + A_1^{-1}(e^{A_1 DT_s} - I)B_1 U$$

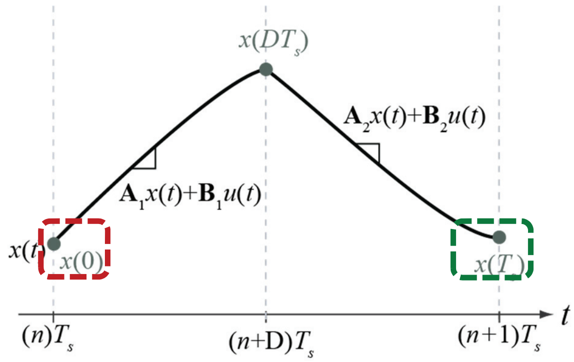
Application to Switching Converter



$$x(DT_s) = e^{A_1 DT_s} x(0) + A_1^{-1}(e^{A_1 DT_s} - I)B_1 U$$

$$x(T_s) = e^{A_2 D' T_s} x(DT_s) + A_2^{-1}(e^{A_2 D' T_s} - I)B_2 U$$

Application to Switching Converter



$$\boldsymbol{x}(DT_s) = e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

$$\boldsymbol{x}(T_s) = e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{x}(DT_s) + \boldsymbol{A}_2{}^{-1}(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U$$

$$\boldsymbol{x}(T_s) = e^{\boldsymbol{A}_2 D' T_s} e^{\boldsymbol{A}_1 DT_s}\boldsymbol{x}(0) + \boldsymbol{A}_2{}^{-1}(e^{\boldsymbol{A}_2 D' T_s} - \boldsymbol{I})\boldsymbol{B}_2 U + e^{\boldsymbol{A}_2 D' T_s}\boldsymbol{A}_1{}^{-1}(e^{\boldsymbol{A}_1 DT_s} - \boldsymbol{I})\boldsymbol{B}_1 U$$

General Form

Generally, for n_{sw} separate switching positions

$$\boldsymbol{x}(T_s) = \left(\prod_{i=n_{sw}}^{1} e^{\boldsymbol{A}_i t_i}\right)\boldsymbol{x}(0) + \sum_{i=1}^{n_{sw}}\left\{\left(\prod_{k=n_{sw}}^{i+1} e^{\boldsymbol{A}_k t_k}\right)\boldsymbol{A}_i{}^{-1}(e^{\boldsymbol{A}_i t_i} - \boldsymbol{I})\boldsymbol{B}_i\right\}U$$

Equation is in the form of a discrete-time system with

$$\boldsymbol{x}[n+1] = \boldsymbol{\Phi}\boldsymbol{x}[n] + \boldsymbol{\Psi}U[n]$$

Again, the effect of changing modulation (i.e. t_i) is hidden in nonlinear terms

$$\hat{\boldsymbol{x}}[n+1] = \boldsymbol{\Phi}\hat{\boldsymbol{x}}[n] + \boldsymbol{\Psi}\hat{u}[n] + \boldsymbol{\Gamma}\hat{d}[n]$$

Find $\boldsymbol{\Gamma}$ by small-signal modeling