Review: Projection

Projection in R^n / C^n

- P : Orthogonal projection of u into span$\{e_1, \dots, e_m\}$, $m \le n$.

Let e_i, i = 1 ... m is **orthonormal** basis, i.e.

$$(e_i, e_j) = 0 \qquad \text{for } i \neq j \text{ and}$$

$$(e_i, e_j) = 1 \qquad \text{for } i = j$$

$$\mathbf{P\,u = \underline{(u, e_1)\,e_1} + \dots + (u, e_m)\,e_m}$$

↑

Orthogonal projection of u on e_1

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What if the basis is not orthonormal?

- We can orthonormalize it. **How?**

  Can get one from every subspace by **Gram-Schmidt** orthogonalization:

  Input : m linearly independent vectors $x_1, \ldots, x_m$

  Output : m orthonormal vectors $x_1, \ldots, x_m$

CGS

1. $x_1 = x_1 / \| x_1 \|$
2. do i = 2, m
3. $\quad x_i = x_i - (x_i, x_1)\, x_1 - \ldots - (x_i, x_{i-1})\, x_{i-1}$
4. $\quad x_i = x_i / \| x_i \|$
5. enddo

MGS

3'. do j = 1, i-1

$\quad x_i = x_i - (x_i, x_j)\, x_j$

enddo

What if the basis is not orthonormal?

- If we do not want to orthonormalize:

$$\mathbf{u} \approx \mathbf{P\,u} = \mathbf{c_1\,x_1 + c_2\,x_2 + \ldots + c_m\,x_m}$$ / 'Multiply' by $x_1, \ldots, x_m$ to get

$$\left|\begin{array}{l} \mathbf{(u\,,x_1) = c_1\,(x_1\,,x_1) + c_2\,(x_2\,,x_1) + \ldots + c_m\,(x_m\,,x_1)} \\ \ldots \\ \mathbf{(u,x_m) = c_1\,(x_1\,,x_m) + c_2\,(x_2\,,x_m) + \ldots + c_m\,(x_m\,,x_m)} \end{array}\right.$$

- These are the so called **Petrov-Galerkin conditions**
- We saw examples of their use in
  * optimization (Homework 6), and
  * PDE discretization, e.g. FEM

Homework #6, Part I, Problem 2

- **Is the following a QR factorization for A?**

  1. $G = A^T A$
  2. $G = L L^T$ (Cholesky factorization)
  3. $Q = A (L^T)^{-1}$

- From (3), assuming the operations used make sense, we have

  $A = Q L^T$ here L^T is upper triangular, so all is left is check if Q is such that $\mathbf{Q^T Q = I}$?

$$Q^T Q \overset{(3)}{=} (A (L^T)^{-1})^T (A (L^T)^{-1}) = (L^{-1} \underbrace{A^T) (A}_{\text{Use (1) and (2) to replace it by } L L^T} L^{-T}) = L^{-1} L \; L^T L^{-T} = I$$

Homework #6, Part I, Problem 3

- Find the projection of $f(x) = \sin(x)$ in $V_1 = \text{span}\{x, x^3, x^5\})$ on interval [-1, 1] using inner-product

  $(f, g) = \int_{-1}^{1} f(x)\, g(x)\, dx$ and norm $|| f || = (f,f)^{1/2}$

Approach I

* construct orthonormal basis, e.g. CGS

$y_1 = x \,/\, || x ||$

$y_2 = x^3 - (x^3, y_1)\, y_1$, $\quad y_2 = y_2 \,/\, ||y_2||$

$y_3 = x^5 - (x^5, y_1)y_1 - (x^5, y_2)y_2$, $\quad y_3 = y_3 \,/\, ||y_3||$

$P\, f(x) = (\sin(x), y_1)y_1 + (\sin(x), y_2)y_2 + (\sin(x), y_3)\, y_3$

Approach II

* directly

(1) $\sin(x) \approx Pf = c_1\, x + c_2\, x^3 + c_3 x^5$ / mult. by x, x^3, x^5

$(\sin(x),x) = c_1\, (x,x) + c_2\, (x^3,x) + c_3(x^5,x)$

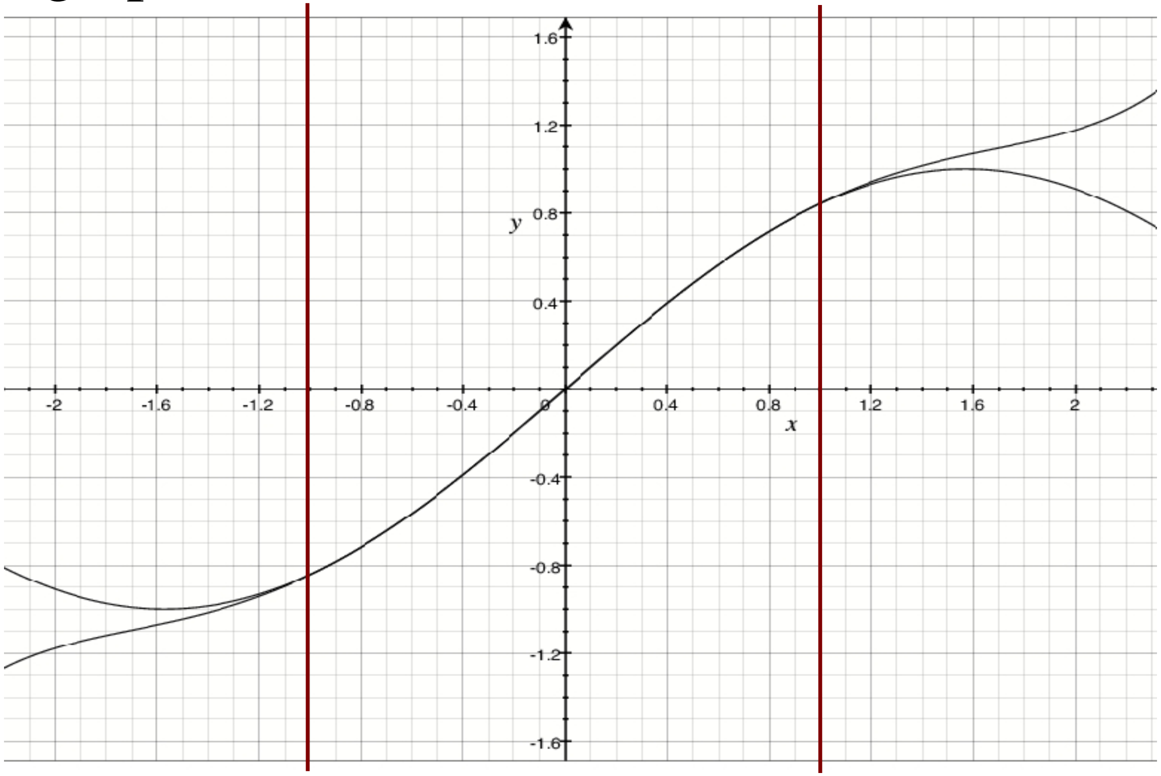
$(\sin(x),x^3) = c_1\, (x,x^3) + c_2\, (x^3,x^3) + c_3(x^5,x^3)$

$(\sin(x),x^5) = c_1\, (x,x^5) + c_2\, (x^3,x^5) + c_3(x^5,x^5)$

solve this 3x3 system and plug c_1, c_2, c_3 back in (1)

Homework #6, Problem 3

(graph from Daniel Lucio)



Pf(x)

f(x) = sin(x)