Chapter 4

HW7:P1

Sections 4-2: Charge and Current Distributions

Problem 4.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2e^{-2z}$ (mC/m^3).

Solution: For the cube shown in Fig. P4.1, application of Eq. (4.5) gives

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} xy^2 e^{-2z} \, dx \, dy \, dz$$

$$= \left(\frac{-1}{12} x^2 y^3 e^{-2z}\right)\Bigg|_{x=0}^{2}\Bigg|_{y=0}^{2}\Bigg|_{z=0}^{2} = \frac{8}{3}(1 - e^{-4}) = 2.62 \text{ mC}.$$

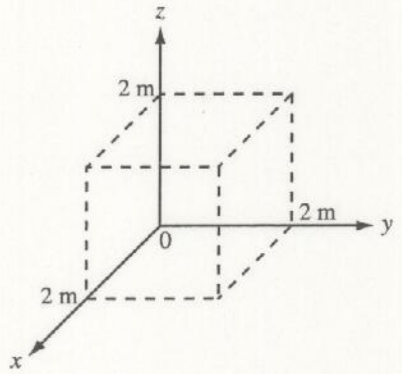


Figure P4.1: Cube of Problem 4.1.

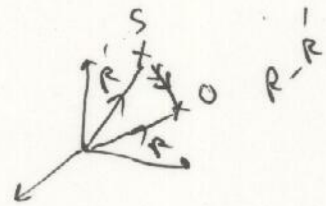
HW7:P2

Problem 4.2 Find the total charge contained in a cylindrical volume defined by $r \leq 2$ m and $0 \leq z \leq 3$ m if $\rho_v = 20rz$ (mC/m^3).

Solution: For the cylinder shown in Fig. P4.2, application of Eq. (4.5) gives

$$Q = \int_{z=0}^{3} \int_{\phi=0}^{2\pi} \int_{r=0}^{2} 20rz \, r \, dr \, d\phi \, dz$$

$$= \left(\frac{10}{3} r^3 \phi z^2\right)\Bigg|_{r=0}^{2}\Bigg|_{\phi=0}^{2\pi}\Bigg|_{z=0}^{3} = 480\pi \text{ (mC)} = 1.5 \text{ C}.$$



$$|R| = \sqrt{1^2+1^2+5^2} = \sqrt{27}.$$

$$\begin{aligned}\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3}+\frac{\mathbf{R}_2}{|\mathbf{R}|^3}+\frac{\mathbf{R}_3}{|\mathbf{R}|^3}+\frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right]\\ &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{-\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}\right]\\ &= \hat{\mathbf{z}}\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\frac{5\times 40\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{1.42}{\pi\varepsilon_0}\times 10^{-6}\ \text{(V/m)} = \hat{\mathbf{z}}51.2\ \text{(kV/m)}.\end{aligned}$$

HW7:P3 →

Problem 4.10 Three point charges, each with $q = 3$ nC, are located at the corners of a triangle in the x–y plane, with one corner at the origin, another at $(2\ \text{cm}, 0, 0)$, and the third at $(0, 2\ \text{cm}, 0)$. Find the force acting on the charge located at the origin.

Solution: Use Eq. (4.19) to determine the electric field at the origin due to the other two point charges [Fig. P4.10]:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\left[\frac{3\ \text{nC}\ (-\hat{\mathbf{x}}0.02)}{(0.02)^3}\right] + \frac{3\ \text{nC}\ (-\hat{\mathbf{y}}0.02)}{(0.02)^3} = -67.4(\hat{\mathbf{x}}+\hat{\mathbf{y}})\ \text{(kV/m) at } \mathbf{R} = 0.$$

Employ Eq. (4.14) to find the force $\mathbf{F} = q\mathbf{E} = -202.2(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ (μN).

You don't need to follow the procedure here.
You should be able to solve the problem
in a much simpler, faster, more intuitive way.
See the example in the course note.

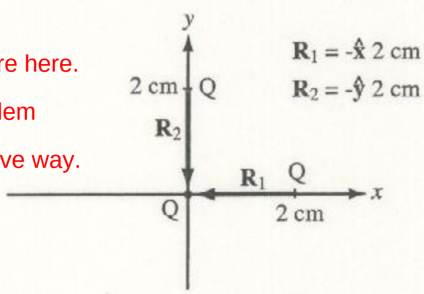


Figure P4.10: Locations of charges in Problem 4.10.

HW7:P4

Problem 4.11 Charge $q_1 = 6\ \mu$C is located at $(1\ \text{cm}, 1\ \text{cm}, 0)$ and charge q_2 is located at $(0, 0, 4\ \text{cm})$. What should q_2 be so that $\mathbf{E}$ at $(0, 2\ \text{cm}, 0)$ has no y-component?

Solution: For the configuration of Fig. P4.11, use of Eq. (4.19) gives

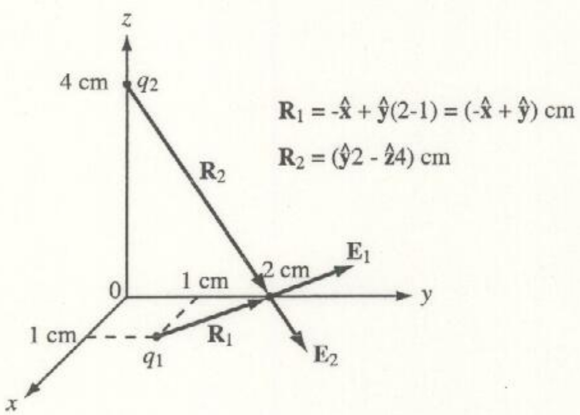


Figure P4.11: Locations of charges in Problem 4.11.

$$\begin{aligned}\mathbf{E}(\mathbf{R} = \hat{\mathbf{y}}2\text{cm}) &= \frac{1}{4\pi\varepsilon}\left[\frac{6\mu\text{C}(-\hat{\mathbf{x}}+\hat{\mathbf{y}}) \times 10^{-2}}{(2 \times 10^{-2})^{3/2}} + \frac{q_2(\hat{\mathbf{y}}2 - \hat{\mathbf{z}}4) \times 10^{-2}}{(20 \times 10^{-2})^{3/2}}\right] \\ &= \frac{1}{4\pi\varepsilon}[-\hat{\mathbf{x}}21.21 \times 10^{-6} + \hat{\mathbf{y}}(21.21 \times 10^{-6} + 0.224q_2) \\ &\quad - \hat{\mathbf{z}}0.447q_2] \quad \text{(V/m)}.\end{aligned}$$

If $E_y = 0$, then $q_2 = -21.21 \times 10^{-6}/0.224 \approx -94.69$ (μC).

Problem 4.12 A line of charge with uniform density $\rho_l = 8$ (μC/m) exists in air along the z-axis between $z = 0$ and $z = 5$ cm. Find $\mathbf{E}$ at (0,10 cm,0).

R = (0, 0.1, 0)
R' = (0, 0, z)

Solution: Use of Eq. (4.21c) for the line of charge shown in Fig. P4.12 gives

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_l\, dl'}{R'^2}, \\ R' &= \hat{\mathbf{y}}0.1 - \hat{\mathbf{z}}z \\ &= \frac{1}{4\pi\varepsilon_0}\int_{z=0}^{0.05} (8 \times 10^{-6}) \frac{(\hat{\mathbf{y}}0.1 - \hat{\mathbf{z}}z)}{[(0.1)^2 + z^2]^{3/2}}\, dz \\ &= \frac{8 \times 10^{-6}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{y}}10z + \hat{\mathbf{z}}}{\sqrt{(0.1)^2 + z^2}}\right]\Bigg|_{z=0}^{0.05} \\ &= 71.86 \times 10^3\,[\hat{\mathbf{y}}4.47 - \hat{\mathbf{z}}1.06] = \hat{\mathbf{y}}321.4 \times 10^3 - \hat{\mathbf{z}}76.2 \times 10^3 \quad \text{(V/m)}.\end{aligned}$$

With $R = h/\cos\theta$, we integrate from $y = 0$ to $d/2$, which corresponds to $\theta = 0$ to $\theta_0 = \sin^{-1}[(d/2)/(h^2 + (d/2)^2)^{1/2}]$. Thus,

$$\mathbf{E} = \int_0^{d/2} d\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{d/2}\frac{\cos\theta}{R}\,dy = \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{\theta_0}\frac{\cos^2\theta}{h}\cdot\frac{h}{\cos^2\theta}\,d\theta$$
$$= \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\theta_0.$$

For an infinitely wide sheet, $\theta_0 = \pi/2$ and $\mathbf{E} = \hat{\mathbf{z}}\dfrac{\rho_s}{2\varepsilon_0}$, which is identical with Eq. (4.25).

HW7:P5 → **Problem 4.20** Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2),$$

determine

(a) ρ_v by applying Eq. (4.26),

(b) the total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin, and

(c) the total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathcal{V}}\nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} 0\,dx\,dy\,dz = 0.$$

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$Q = \oint\mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}},$$

$$F_{\text{front}} = \int_{y=0}^{2}\int_{z=0}^{2}\left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=2}\cdot(\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=2}\,dz\,dy = \left(2z\left(2y+\frac{1}{2}y^2\right)\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = 24,$$

$$F_{\text{back}} = \int_{y=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{x=0} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= -\int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=0}\,dz\,dy = -\left(zy^2\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = -8,$$

$$F_{\text{right}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{y=2} \cdot (\hat{\mathbf{y}}\,dz\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=2}\,dz\,dx = \left(z\left(\frac{3}{2}x^2 - 4x\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{y=0} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=0}\,dz\,dx = -\left(z\left(\frac{3}{2}x^2\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{z=2} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=2}\,dy\,dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{z=0} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=0}\,dy\,dx = 0.$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 24 - 8 - 4 - 12 + 0 + 0 = 0.$

HW7:P5 continued

Problem 4.21 Repeat Problem 4.20 for $\mathbf{D} = \hat{\mathbf{x}}xy^3z^3$ (C/m^2).

Solution:

(a) From Eq. (4.26), $\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(xy^3z^3) = y^3z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla \cdot \mathbf{D}\,d\mathcal{V} = \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2} y^3z^3\,dx\,dy\,dz = \frac{xy^4z^4}{16}\Bigg|_{x=0}^{2}\Bigg|_{y=0}^{2}\Bigg|_{z=0}^{2} = 32 \text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}}D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$F_{\text{front}} = \int_{z=0}^{2}\int_{y=0}^{2} (\hat{\mathbf{x}}xy^3z^3)\bigg|_{x=2} \cdot (\hat{\mathbf{x}}\,dy\,dz)$$
$$= \int_{z=0}^{2}\int_{y=0}^{2} xy^3z^3\bigg|_{x=2} dy\,dz = \left(2\left(\frac{y^4z^4}{16}\right)\bigg|_{y=0}^{2}\right)\bigg|_{z=0}^{2} = 32,$$
$$F_{\text{back}} = \int_{z=0}^{2}\int_{y=0}^{2} (\hat{\mathbf{x}}xy^3z^3)\bigg|_{x=0} \cdot (-\hat{\mathbf{x}}\,dy\,dz) = -\int_{z=0}^{2}\int_{y=0}^{2} xy^3z^3\bigg|_{x=0} dy\,dz = 0.$$

Thus $Q = \oint \mathbf{D}\cdot d\mathbf{s} = 32+0+0+0+0+0 = 32$ C.

Problem 4.22 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to find $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}}D_{\text{R}}$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}}R^2 \sin\theta\, d\theta\, d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S. For a spherical surface of radius R,

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\hat{\mathbf{R}}D_{\text{R}})\cdot(\hat{\mathbf{R}}R^2\sin\theta\, d\theta\, d\phi) = Q_{\text{tot}},$$
$$D_{\text{R}}R^2(2\pi)[-\cos\theta]_0^{\pi} = Q_{\text{tot}},$$
$$D_{\text{R}} = \frac{Q_{\text{tot}}}{4\pi R^2}.$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon\mathbf{E}$. Thus, we find $\mathbf{E}$ from $\mathbf{D}$.