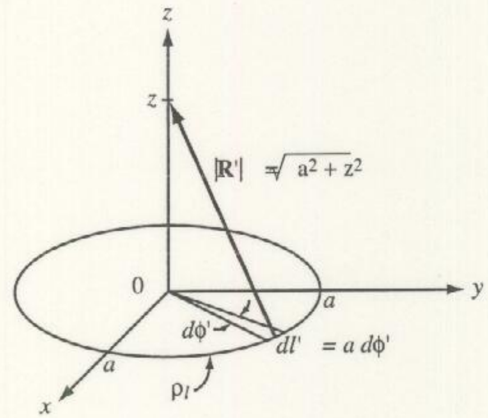


Figure P4.29: Ring of charge.

(b) From Eq. (4.51),

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{\partial}{\partial z}(a^2+z^2)^{-1/2} = \hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{z}{(a^2+z^2)^{3/2}} \quad \text{(V/m)}.$$

Problem 4.30 Show that the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_l along the z-axis is $V_{12} = (\rho_l/2\pi\varepsilon_0)\ln(r_2/r_1)$.

Solution: From Eq. (4.33), the electric field due to an infinite line of charge is

$$\mathbf{E} = \hat{\mathbf{r}}E_r = \hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon_0 r}.$$

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1}\mathbf{E}\cdot d\mathbf{l} = -\int_{r_2}^{r_1}\frac{\hat{\mathbf{r}}\rho_l}{2\pi\varepsilon_0 r}\cdot\hat{\mathbf{r}}\,dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right).$$

Problem 4.31 Find the electric potential V at a location a distance b from the origin in the x–y plane due to a line charge with charge density ρ_l and of length l. The line charge is coincident with the z-axis and extends from $z = -l/2$ to $z = l/2$.

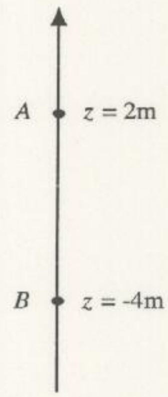


Figure P4.34: Potential between B and A.

HW9:P1

Problem 4.35 An infinitely long line of charge with uniform density $\rho_l = 9$ (nC/m) lies in the x–y plane parallel to the y-axis at $x = 2$ m. Find the potential V_{AB} at point $A(3\text{ m}, 0, 4\text{ m})$ in Cartesian coordinates with respect to point $B(0,0,0)$ by applying the result of Problem 4.30.

Solution: According to Problem 4.30,

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

where r_1 and r_2 are the distances of A and B. In this case,

$$r_1 = \sqrt{(3-2)^2 + 4^2} = \sqrt{17}\text{ m},$$
$$r_2 = 2\text{ m}.$$

Hence,

$$V_{AB} = \frac{9 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}} \ln\left(\frac{2}{\sqrt{17}}\right) = -117.09\text{ V}.$$

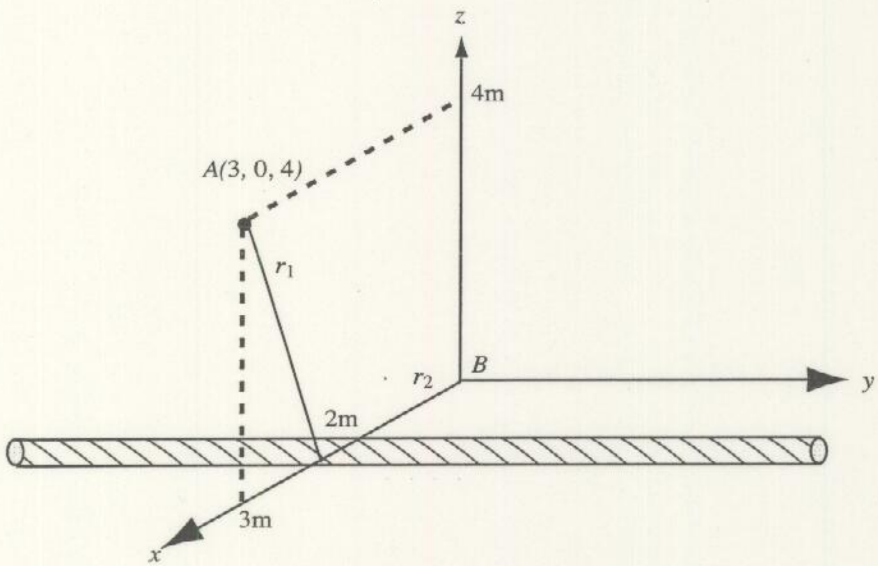


Figure P4.35: Line of charge parallel to y-axis.

Problem 4.36 The x–y plane contains a uniform sheet of charge with $\rho_{s_1} = 0.2$ (nC/m^2) and a second sheet with $\rho_{s_2} = -0.2$ (nC/m^2) occupies the plane $z = 6$ m. Find V_{AB}, V_{BC}, and V_{AC} for $A(0,0,6\text{ m})$, $B(0,0,0)$, and $C(0,-2\text{ m},2\text{ m})$.

Solution: We start by finding the **E** field in the region between the plates. For any point above the x–y plane, $\mathbf{E}_1$ due to the charge on x–y plane is, from Eq. (4.25),

$$\mathbf{E}_1 = \hat{\mathbf{z}}\frac{\rho_{s_1}}{2\varepsilon_0}.$$

In the region below the top plate, **E** would point downwards for positive ρ_{s_2} on the top plate. In this case, $\rho_{s_2} = -\rho_{s_1}$. Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{z}}\frac{\rho_{s_1}}{2\varepsilon_0} - \hat{\mathbf{z}}\frac{\rho_{s_2}}{2\varepsilon_0} = \hat{\mathbf{z}}\frac{2\rho_{s_1}}{2\varepsilon_0} = \hat{\mathbf{z}}\frac{\rho_{s_1}}{\varepsilon_0}.$$

Since **E** is along $\hat{\mathbf{z}}$, only change in position along z can result in change in voltage.

$$V_{AB} = -\int_0^6 \hat{\mathbf{z}}\frac{\rho_{s_1}}{\varepsilon_0}\cdot\hat{\mathbf{z}}\,dz = -\frac{\rho_{s_1}}{\varepsilon_0}z\bigg|_0^6 = -\frac{6\rho_{s_1}}{\varepsilon_0} = -\frac{6\times 0.2\times 10^{-9}}{8.85\times 10^{-12}} = -135.59\text{ V}.$$

Section 4-9: Boundary Conditions

HW9:P2

HW9.

Problem 4.43 With reference to Fig. 4-19, find $\mathbf{E}_1$ if $\mathbf{E}_2 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}2$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$, and the boundary has a surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m^2). What angle does $\mathbf{E}_2$ make with the z-axis?

Solution: We know that $\mathbf{E}_{1t} = \mathbf{E}_{2t}$ for any 2 media. Hence, $\mathbf{E}_{1t} = \mathbf{E}_{2t} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$. Also, $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_s$ (from Table 4.3). Hence, $\varepsilon_1(\mathbf{E}_1 \cdot \hat{\mathbf{n}}) - \varepsilon_2(\mathbf{E}_2 \cdot \hat{\mathbf{n}}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1} = \frac{3.54 \times 10^{-11}}{2\varepsilon_0} + \frac{18(2)}{2} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \quad \text{(V/m)}.$$

Hence, $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}20$ (V/m). Finding the angle $\mathbf{E}_2$ makes with the z-axis:

$$\mathbf{E}_2 \cdot \hat{\mathbf{z}} = |\mathbf{E}_2| \cos\theta, \qquad 2 = \sqrt{9+4+4}\cos\theta, \qquad \theta = \cos^{-1}\left(\frac{2}{\sqrt{17}}\right) = 61^\circ.$$

Problem 4.44 An infinitely long conducting cylinder of radius a has a surface charge density ρ_s. The cylinder is surrounded by a dielectric medium with $\varepsilon_r = 4$ and contains no free charges. If the tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\boldsymbol{\phi}} \cos^2\phi / r^2$, find ρ_s.

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$\mathbf{E}_2 = \hat{\mathbf{r}} E_r - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos^2\phi,$$

with E_r, the normal component of $\mathbf{E}_2$, unknown. The surface charge density is related to E_r. To find E_r, we invoke Gauss's law in medium 2:

$$\nabla \cdot \mathbf{D}_2 = 0,$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(-\frac{1}{r^2}\cos^2\phi\right) = 0,$$

which leads to

$$\frac{\partial}{\partial r}(rE_r) = \frac{\partial}{\partial \phi}\left(\frac{1}{r^2}\cos^2\phi\right) = -\frac{2}{r^2}\sin\phi\cos\phi.$$

Integrating both sides with respect to r,

$$\int \frac{\partial}{\partial r}(rE_r)\, dr = -2\sin\phi\cos\phi \int \frac{1}{r^2}\, dr$$

$$rE_r = \frac{2}{r}\sin\phi\cos\phi,$$

Solution: From Eq. (4.131),

$$\mathbf{F} = -\hat{\mathbf{z}}\frac{\varepsilon A|\mathbf{E}|^2}{2} = -\hat{\mathbf{z}}2\varepsilon_0(5\times10^{-4})\left(\frac{50}{0.02}\right)^2 = -\hat{\mathbf{z}}55.3\times10^{-9} \quad \text{(N)}.$$

HW9:P3 HW9

Problem 4.49 Dielectric breakdown occurs in a material whenever the magnitude of the field **E** exceeds the dielectric strength anywhere in that material. In the coaxial capacitor of Example 4-12,

(a) At what value of r is $|E|$ maximum?

(b) What is the breakdown voltage if $a = 1$ cm, $b = 2$ cm, and the dielectric material is mica with $\varepsilon_r = 6$?

Solution:

(a) From Eq. (4.114), $\mathbf{E} = -\hat{\mathbf{r}}\rho_l/2\pi\varepsilon r$ for $a < r < b$. Thus, it is evident that $|\mathbf{E}|$ is maximum at $r = a$.

(b) The dielectric breaks down when $|\mathbf{E}| > 200$ (MV/m) (see Table 4-2), or

$$|\mathbf{E}| = \frac{\rho_l}{2\pi\varepsilon r} = \frac{\rho_l}{2\pi(6\varepsilon_0)(10^{-2})} = 200 \quad \text{(MV/m)},$$

which gives $\rho_l = (200 \text{ MV/m})(2\pi)6(8.854\times10^{-12})(0.01) = 667.6$ (μC/m).

From Eq. (4.115), we can find the voltage corresponding to that charge density,

$$V = \frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right) = \frac{(667.6\,\mu\text{C/m})}{12\pi(8.854\times10^{-12}\text{ F/m})}\ln(2) = 1.39 \quad \text{(MV)}.$$

Thus, $V = 1.39$ (MV) is the breakdown voltage for this capacitor.

Problem 4.50 An electron with charge $Q_e = -1.6\times10^{-19}$ C and mass $m_e = 9.1\times10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area Fig. 4-33 (P4.50). If the voltage across the capacitor is 10 V, find

(a) the force acting on the electron,

(b) the acceleration of the electron, and

(c) the time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

Solution:

(a) The electric force acting on a charge Q_e is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_eE = Q_e\frac{V}{d} = -1.6\times10^{-19}\frac{10}{0.01} = -1.6\times10^{-16} \quad \text{(N)}.$$

Solution: Electrostatic potential energy is given by Eq. (4.124),

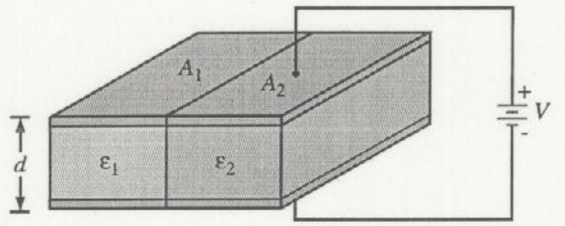
$$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon |\mathbf{E}|^2 \, d\mathcal{V} = \frac{\varepsilon}{2}\int_{z=0}^{3}\int_{y=0}^{2}\int_{x=-1}^{1} [(x^2+2z)^2 + x^4 + (y+z)^2] \, dx\, dy\, dz$$

$$= \frac{4\varepsilon_0}{2}\left(\left(\left(\frac{2}{5}x^5yz + \frac{2}{3}z^2x^3y + \frac{4}{3}z^3xy + \frac{1}{12}(y+z)^4x\right)\Bigg|_{x=-1}^{1}\right)\Bigg|_{y=0}^{2}\right)\Bigg|_{z=0}^{3}$$

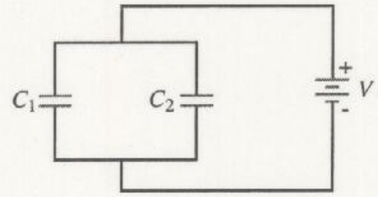
$$= \frac{4\varepsilon_0}{2}\left(\frac{1304}{5}\right) = 4.62 \times 10^{-9} \quad \text{(J)}.$$

HW9:P4

Problem 4.52 Figure 4-34a (P4.52(a)) depicts a capacitor consisting of two parallel, conducting plates separated by a distance d. The space between the plates



(a)



(b)

Figure P4.52: (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

contains two adjacent dielectrics, one with permittivity ε_1 and surface area A_1 and another with ε_2 and A_2. The objective of this problem is to show that the capacitance C of the configuration shown in Fig. 4-34a (P4.52(a)) is equivalent to two capacitances in parallel, as illustrated in Fig. 4-34b (P4.52(b)), with

$$C = C_1 + C_2, \tag{4.132}$$

Note: This result is actually an approximation.

Think: What is the condition for this approximation? What was ignored?

By ignoring this, what physics do we violate?

where

$$C_1 = \frac{\varepsilon_1 A_1}{d}, \tag{4.133}$$

$$C_2 = \frac{\varepsilon_2 A_2}{d}. \tag{4.134}$$

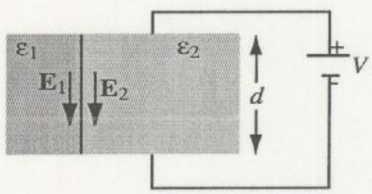
To this end, you are asked to proceed as follows:

(a) Find the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two dielectric layers.

(b) Calculate the energy stored in each section and use the result to calculate C_1 and C_2.

(c) Use the total energy stored in the capacitor to obtain an expression for C. Show that Eq. (4.132) is indeed a valid result.

Solution:



(c)

Figure P4.52: (c) Electric field inside of capacitor.

(a) Within each dielectric section, **E** will point from the plate with positive voltage to the plate with negative voltage, as shown in Fig. P4-52(c). From $V = Ed$,

$$E_1 = E_2 = \frac{V}{d}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathcal{V} = \frac{1}{2}\varepsilon_1 \frac{V^2}{d^2} \cdot A_1 d = \frac{1}{2}\varepsilon_1 V^2 \frac{A_1}{d}.$$

But, from Eq. (4.121),

$$W_{e_1} = \frac{1}{2} C_1 V^2.$$

Hence $C_1 = \varepsilon_1 \dfrac{A_1}{d}$. Similarly, $C_2 = \varepsilon_2 \dfrac{A_2}{d}$.

(c) Total energy is

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2}\frac{V^2}{d}(\varepsilon_1 A_1 + \varepsilon_2 A_2) = \frac{1}{2} C V^2.$$

Hence,

$$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2.$$

HW9:P4

continued

HW9

Problem 4.53 Use the result of Problem 4.52 to determine the capacitance for each of the following configurations:

(a) conducting plates are on top and bottom faces of rectangular structure in Fig. 4-35(a) (P4.53(a)),

(b) conducting plates are on front and back faces of structure in Fig. 4-35(a) (P4.53(a)),

(c) conducting plates are on top and bottom faces of the cylindrical structure in Fig. 4-35(b) (P4.53(b)).

Solution:

(a) The two capacitors share the same voltage; hence they are in parallel.

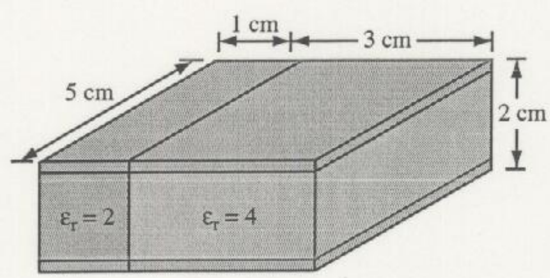
$$C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(5\times 1)\times 10^{-4}}{2\times 10^{-2}} = 5\varepsilon_0 \times 10^{-2},$$
$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(5\times 3)\times 10^{-4}}{2\times 10^{-2}} = 30\varepsilon_0 \times 10^{-2},$$
$$C = C_1 + C_2 = (5\varepsilon_0 + 30\varepsilon_0)\times 10^{-2} = 0.35\varepsilon_0 = 3.1\times 10^{-12}\text{ F}.$$

(b)

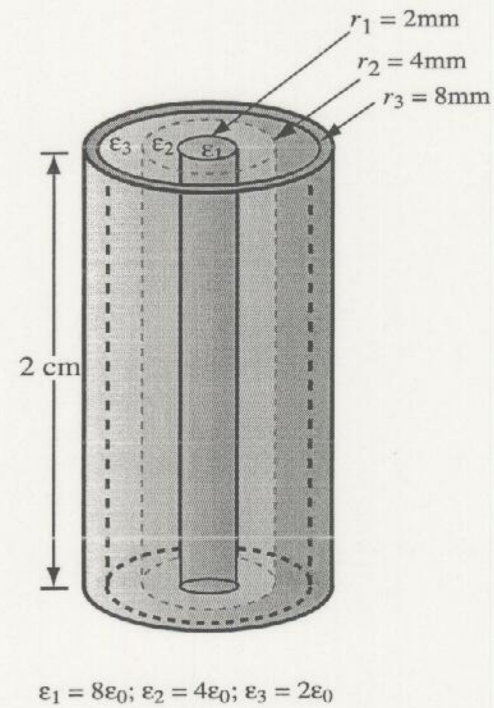
$$C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(2\times 1)\times 10^{-4}}{5\times 10^{-2}} = 0.8\,\varepsilon_0 \times 10^{-2},$$
$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(3\times 2)\times 10^{-4}}{5\times 10^{-2}} = \frac{24}{5}\varepsilon_0 \times 10^{-2},$$
$$C = C_1 + C_2 = 0.5\times 10^{-12}\text{ F}.$$

(c)

$$C_1 = \varepsilon_1 \frac{A_1}{d} = 8\varepsilon_0 \frac{(\pi r_1^2)}{2\times 10^{-2}} = \frac{4\pi\varepsilon_0}{10^{-2}}(2\times 10^{-3})^2 = 0.04\times 10^{-12}\text{ F},$$
$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(\pi(r_2^2 - r_1^2))}{2\times 10^{-2}} = \frac{2\pi\varepsilon_0}{10^{-2}}[(4\times 10^{-3})^2 - (2\times 10^{-3})^2] = 0.06\times 10^{-12}\text{ F},$$
$$C_3 = \varepsilon_3 \frac{A_3}{d} = 2\varepsilon_0 \frac{(\pi(r_3^2 - r_2^2))}{2\times 10^{-2}} = \frac{\pi\varepsilon_0}{10^{-2}}[(8\times 10^{-3})^2 - (4\times 10^{-3})^2] = 0.12\times 10^{-12}\text{ F},$$
$$C = C_1 + C_2 + C_3 = 0.22\times 10^{-12}\text{ F}.$$



(a)



$\varepsilon_1 = 8\varepsilon_0$; $\varepsilon_2 = 4\varepsilon_0$; $\varepsilon_3 = 2\varepsilon_0$

(b)

Figure P4.53: Dielectric sections for Problems 4.53 and 4.55.

HW9:P5

Problem 4.54 The capacitor shown in Fig. 4-36 (P4.54) consists of two parallel dielectric layers. We wish to use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2}, \tag{4.136}$$

where

$$C_1 = \varepsilon_1 \frac{A}{d_1}, \qquad C_2 = \varepsilon_2 \frac{A}{d_2}.$$

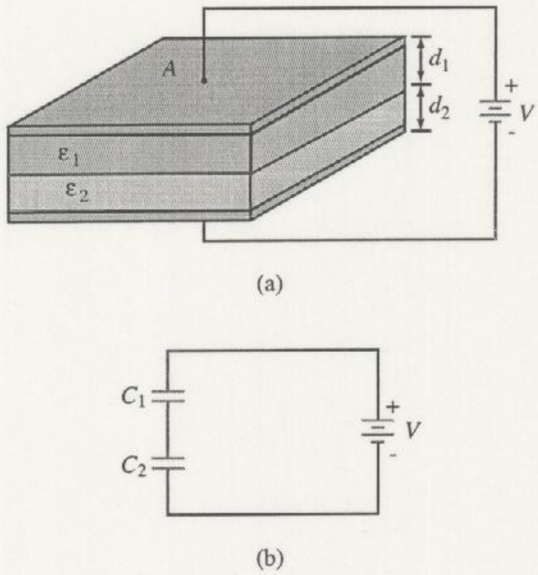


Figure P4.54: (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.54).

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ε_1, ε_2, V, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

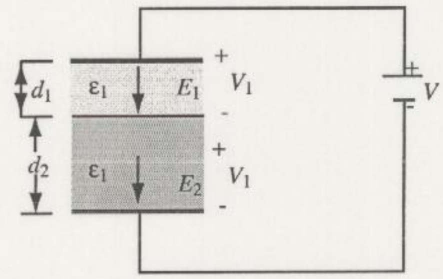


Figure P4.54: (c) Electric fields inside of capacitor.

(c) Show that C is given by Eq. (4.136).

Solution:

(a) If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \qquad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of **D** is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2,$$

which can be solved for E_1:

$$E_1 = \frac{V}{d_1 + \dfrac{\varepsilon_1}{\varepsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \dfrac{\varepsilon_2}{\varepsilon_1} d_1}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathcal{V}_1 = \frac{1}{2}\varepsilon_1 \left(\frac{V}{d_1 + \dfrac{\varepsilon_1}{\varepsilon_2} d_2}\right)^2 \cdot Ad_1 = \frac{1}{2}V^2 \left[\frac{\varepsilon_1\varepsilon_2^2 Ad_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right],$$

$$W_{e_2} = \frac{1}{2}\varepsilon_2 E_2^2 \cdot \mathcal{V}_2 = \frac{1}{2}\varepsilon_2 \left(\frac{V}{d_2 + \dfrac{\varepsilon_2}{\varepsilon_1} d_1}\right)^2 \cdot Ad_2 = \frac{1}{2}V^2 \left[\frac{\varepsilon_1^2\varepsilon_2 Ad_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right],$$

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2}V^2 \left[\frac{\varepsilon_1\varepsilon_2^2 Ad_1 + \varepsilon_1^2\varepsilon_2 Ad_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right].$$

But $W_e = \frac{1}{2}CV^2$, hence,

$$C = \frac{\varepsilon_1\varepsilon_2^2 Ad_1 + \varepsilon_1^2\varepsilon_2 Ad_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \varepsilon_1\varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1\varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for C by $A/d_1 d_2$, we have

$$C = \frac{\dfrac{\varepsilon_1 A}{d_1} \cdot \dfrac{\varepsilon_2 A}{d_2}}{\dfrac{\varepsilon_1 A}{d_1} + \dfrac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \qquad C_2 = \frac{\varepsilon_2 A}{d_2}.$$

HW9:P5 continued

HW 9.

Problem 4.55 Use the expressions given in Problem 4.54 to determine the capacitance for the configurations in Fig. 4.35(a) (P4.55) when the conducting plates are placed on the right and left faces of the structure.

Solution:

$$C_1 = \varepsilon_1 \frac{A}{d_1} = 2\varepsilon_0 \frac{(2\times 5)\times 10^{-4}}{1\times 10^{-2}} = 20\varepsilon_0 \times 10^{-2} = 1.77\times 10^{-12}\text{ F},$$

$$C_2 = \varepsilon_2 \frac{A}{d_2} = 4\varepsilon_0 \frac{(2\times 5)\times 10^{-4}}{3\times 10^{-2}} = 1.18\times 10^{-12}\text{ F},$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1.77\times 1.18}{1.77 + 1.18} \times 10^{-12} = 0.71\times 10^{-12}\text{ F}.$$

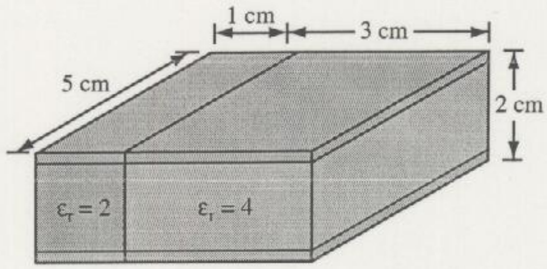


Figure P4.55: Dielectric section for Problem 4.55.

Section 4-12: Image Method

HW9:P6

Problem 4.56 With reference to Fig. 4-37 (P4.56), charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent), and

(b) then find the electric potential and electric field at an arbitrary point $P(0, y, z)$.

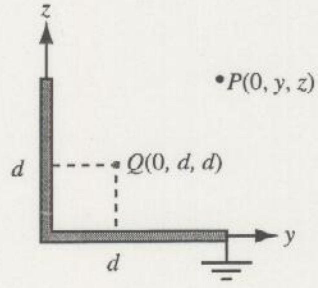


Figure P4.56: Charge Q next to two perpendicular, grounded, conducting half planes.

Solution:

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location

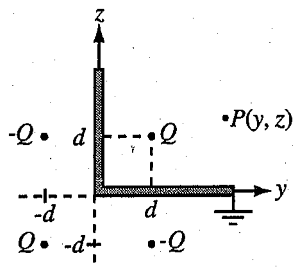


Figure P4.56: (a) Image charges.

$(0, d, -d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0, -d, d)$ and $+Q$ at $(0, -d, -d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$\begin{aligned}
V(x,y,z) &= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)|} - \frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)|}\right.\\
&\quad \left. + \frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)|} - \frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)|}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right.\\
&\quad \left. + \frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}} - \frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+y^2-2yd+z^2-2zd+2d^2}}\right.\\
&\quad - \frac{1}{\sqrt{x^2+y^2+2yd+z^2-2zd+2d^2}}\\
&\quad + \frac{1}{\sqrt{x^2+y^2+2yd+z^2+2zd+2d^2}}\\
&\quad \left. - \frac{1}{\sqrt{x^2+y^2-2yd+z^2+2zd+2d^2}}\right) \quad \text{(V)}.
\end{aligned}$$

From Eq. (4.51),

$$\begin{aligned}
\mathbf{E} &= -\nabla V \\
&= \frac{Q}{4\pi\varepsilon}\left(\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}} - \nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right. \\
&\qquad \left. + \nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}} - \nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right) \\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y-d)^2+(z-d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y+d)^2+(z-d)^2)^{3/2}}\right. \\
&\qquad \left. + \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y+d)^2+(z+d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y-d)^2+(z+d)^2)^{3/2}}\right) \quad \text{(V/m)}.
\end{aligned}$$

Problem 4.57 Conducting wires above a conducting plane carry currents I_1 and I_2 in the directions shown in Fig. 4-38 (P4.57). Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to I_1 and I_2?

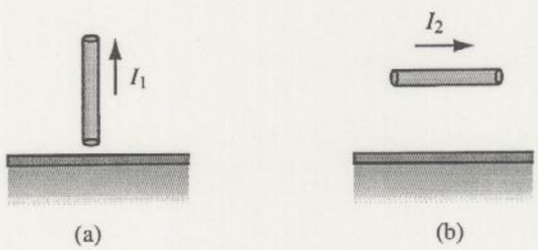


Figure P4.57: Currents above a conducting plane (Problem 4.57).

Solution:

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of I_1 is same as I_1.