The Electrostatic Field Is Conservative

The work done to move $q$ from $a$ to $b$:

$$W = -\int_{a}^{b} \mathbf{F} \cdot d\mathbf{l}$$

$$= -q \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l}$$

Let’s first consider the field of a point charge.

Since $E = E(r)$ (recall Coulomb’s law), $W$ is independent of the path.

Therefore,

$$\int \mathbf{E} \cdot d\mathbf{l} = 0$$

for a point charge.

By superposition, for any electrostatic field,

$$\int \mathbf{E} \cdot d\mathbf{l} = 0$$
We should point out an important fact. For any radial force the work done is independent of the path, and there exists a potential. If you think about it, the entire argument we made above to show that the work integral was independent of the path depended only on the fact that the force from a single charge was radial and spherically symmetric. It did not depend on the fact that the dependence on distance was as $1/r^2$—there could have been any $r$ dependence.

--Richard Feynman
Similarly, the gravitational field is also conservative, due to the similarity between Newton’s law of universal gravitation and Coulomb’s law.

Exceptions are found only in abstract art:

Waterfall by M.C. Escher
Potential.
Consider a charge \( q \) in a field \( \vec{E} \).

\[- \int \vec{F} \cdot d\vec{l} = -q \int \vec{E} \cdot d\vec{l} \]

work done by you

\[-q \int \vec{E} \cdot d\vec{l} \]

Recall that \( \int \vec{E} \cdot d\vec{l} = 0 \) for dc.

Therefore, we say that the \( \vec{E} \) field is conservative.

Which means:

(A non-conservative field does NOT violate conservation of energy.)

and, If you move \( q \) from point 1 to point 2, the work you do doesn't depend on the path.

So, we can define a potential, which is equivalent to height in our earth's gravity field.

\[ qV_2 = qV_1 + q \int \vec{E} \cdot d\vec{l} \] along any path.

The exact values of \( V_1 \) & \( V_2 \) don't really matter, what matters is \( V_2 - V_1 \).

Just like heights, you need a reference.
\[ V_2 - V_1 = -\int \vec{E} \cdot d\vec{l} \]

i.e.

\[ dV = -\vec{E} \cdot d\vec{l} = \nabla V \cdot d\vec{l} \]

Therefore.

\[ \nabla V = -\vec{E} \]

Recall.

\[ \nabla V = \left( \frac{\partial V}{\partial x} \hat{x} + \frac{\partial V}{\partial y} \hat{y} + \frac{\partial V}{\partial z} \hat{z} \right) \]

and \( V \) is a scalar.

\[ \nabla V = \frac{\partial V}{\partial x} \hat{x} + \frac{\partial V}{\partial y} \hat{y} + \frac{\partial V}{\partial z} \hat{z} \]

Read the relevant section in Ch. 3 on your own to appreciate the meaning of the gradient.

Here, I can give you a visual picture.

The gradient is the slope along the direction where the slope is the sharpest/steepest.

Say, you have a hill...


Now, let's work on an example:

First, let's consider the potential due to a point charge.
Example: potential distribution due to a point charge

\[ \vec{E} = \frac{q}{4\pi \varepsilon} \frac{\hat{r}}{r^2} \]

If we go side ways, no work.

Let's consider \( V(\infty) = 0 \).

Move right toward \( q \).

\[ V(R) = V(\infty) \pm \int_{\infty}^{R} \vec{E} \cdot d\vec{r} \]
\[ = \frac{q}{4\pi \varepsilon} \left[ -\frac{1}{r} \right]_{\infty}^{R} \]
\[ = \frac{q}{4\pi \varepsilon} \frac{1}{R} \]

pay attention to the sign

Now, let's introduce the so-called Poisson's \( \varepsilon \).

\[ \nabla \cdot \vec{E} = \frac{P}{\varepsilon} \]
\[ \vec{E} = -\nabla V \]

\[ \nabla V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \]

\[ \nabla^2 V = \nabla \cdot (\nabla V) = \left( \hat{x} \frac{\partial^2 V}{\partial x^2} + \hat{y} \frac{\partial^2 V}{\partial y^2} + \hat{z} \frac{\partial^2 V}{\partial z^2} \right) \]

the Laplacian
\[ = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]

Notice that this is a scalar.
\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{P}{\varepsilon} \]