Magnetostatics

Compare to electrostatics:

- Sources of fields

  $\vec{E}$: Coulomb's law

  $\vec{B}$: Biot - Savart law

- Forces due to the fields.

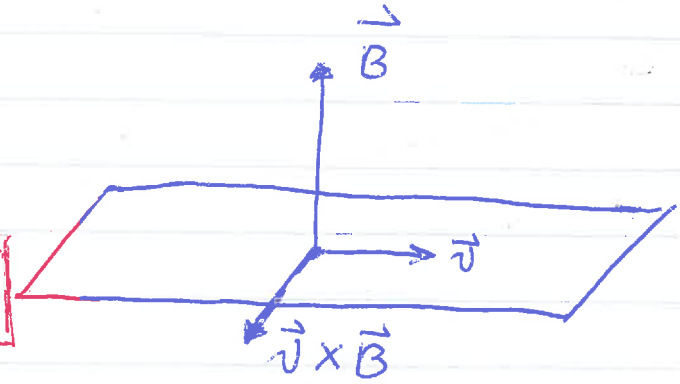
  E : $\vec{F} = q\vec{E}$

  Magnetic :

  $\vec{F} = q\,\vec{v} \times \vec{B}$

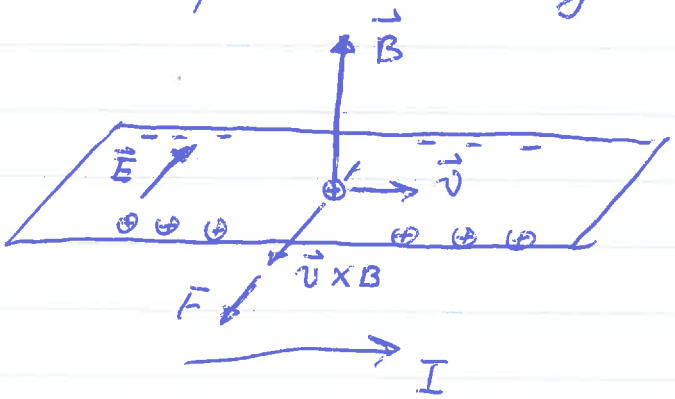
ask question

$\boxed{\text{any work done?}}$



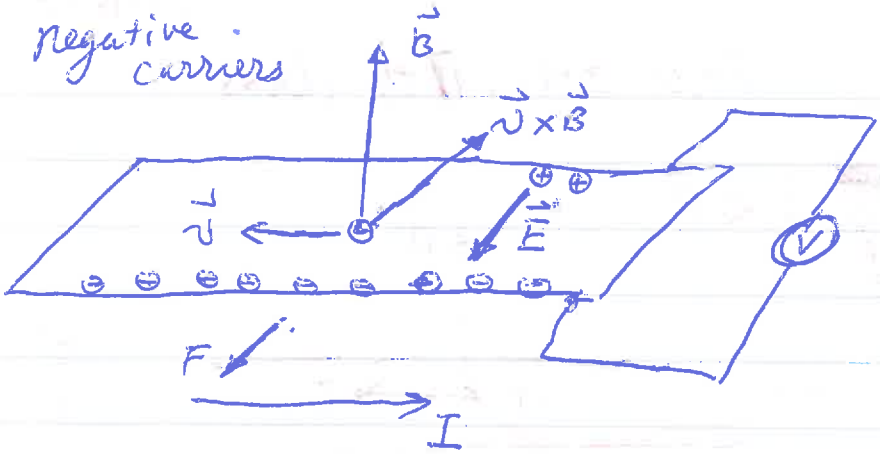
Hall effect

For positive charge carriers



Steady state

$E = vB$



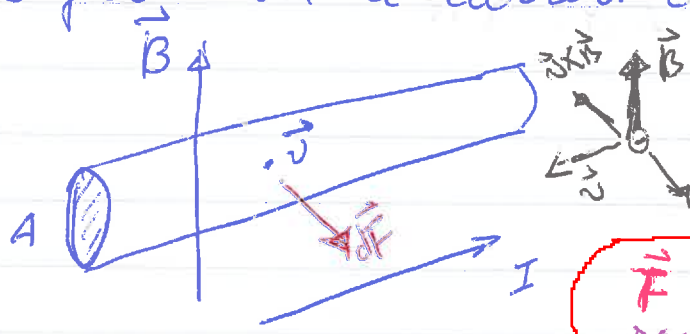
same $\vec{F}$ direction for + & − charge carriers.

By measuring the polarity of the Hall voltage we can determine the sign of the carriers.

One important thing to notice about the magnetic force. It doesn't do any work!

$$F = q\vec{v} \times \vec{B} \perp \vec{v}$$

Magnetic force on a current carrying wire



$I = nqvA$

$\vec{F}$ in same direction regardless of polarity of carriers

q & v can be positive or negative.

For a small piece of the wire $d\vec{l}$.

$$dQ = nqA\,dl$$

$$d\vec{F} = dQ\,\vec{v} \times \vec{B} = (nqA\,dl)\,\vec{v} \times \vec{B}$$

$$= nqvA\,d\vec{l} \times \vec{B} = I\,d\vec{l} \times \vec{B}$$

For a wire from A to B.

$$F = I \int_A^B d\vec{l} \times \vec{B}$$

☆ For a straight wire in uniform B

$$F = I l B$$

If you have a loop.

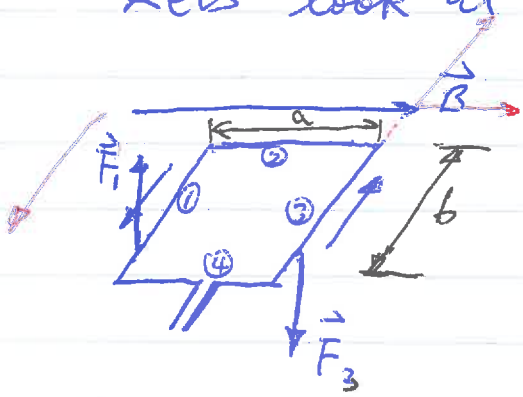
$$F = I \oint d\vec{l} \times \vec{B}$$

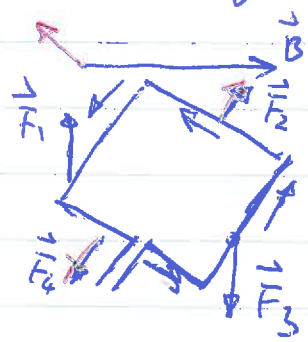
When and only when $\vec{B}$ is a constant

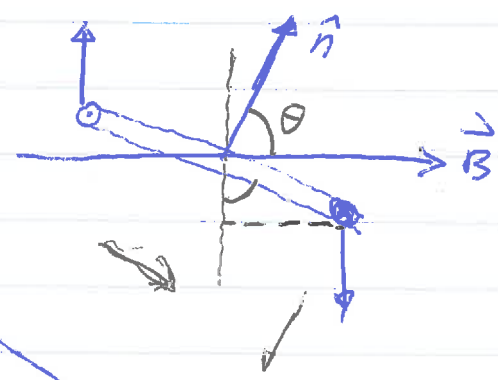
$$F = I\left(\oint d\vec{l}\right) \times \vec{B} = 0$$

$\oint d\vec{l} = 0$ for any loop!

Let's look at a rectangular loop in a $\vec{B}$-field







$\vec{F}_1 + \vec{F}_3 = 0$

$$\begin{cases} \vec{F}_1 + \vec{F}_3 = 0 \\ \vec{F}_2 + \vec{F}_4 = 0 \end{cases}$$

Total force is 0.
Will the loop be set in motion by these forces? — Torque

$$T = F_1 \frac{a}{2} \sin\theta + F_3 \frac{a}{2} \sin\theta$$

$$= F a \sin\theta = I b B a \sin\theta = I A B \sin\theta$$

Define the magnetic moment

$$\vec{m} = N I A \hat{n}$$ # of turns

Then $\vec{T} = \vec{m} \times \vec{B}$

. Vector . direction (right hand rule)
magnitude.

* * *

We derived this equation for a rectangular loop, but it's general

A current-carrying loop is like a magnet: electromagnet.

△ The unit of B

$$F = q\,\vec{v} \times \vec{B}$$

$$[\text{unit of } B] = \frac{N}{C\,\frac{m}{s}} = \frac{N}{A\,m} \equiv T$$

For the $\vec{E}$ field $F = qE$

$\therefore$ vB and E have the same unit

$$[\text{unit of } B] = \frac{V/m}{m/s} = \frac{Vs}{m^2} \equiv T$$

△ We will revisit this later, giving you other forms of the unit T.

We have discussed the force exerted by the magnetic field on a <u>moving</u> charge.

Notice the word <u>moving</u>. ...

Now, question: does a charge gain any energy from a magnetic ~~charge~~ field?

OK, now we discuss the source of the magnetic field.

Again we use the electric field analogy

For the E field, we have Coulomb's law.

$$d\vec{E} = \frac{1}{4\pi\varepsilon}\frac{\rho dV}{R^2}\hat{R} \qquad E = \varepsilon D$$

For the magnetic field, we have the Biot-Savart law.

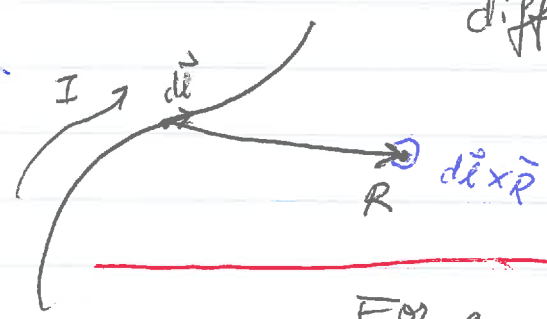
$$d\vec{H} = \frac{1}{4\pi}\frac{I d\vec{l}\times\hat{R}}{R^2} \qquad B = \mu H.$$

Similarities: Both carry French names w/ wierd pronounciations

Both proportional to $1/R^2$

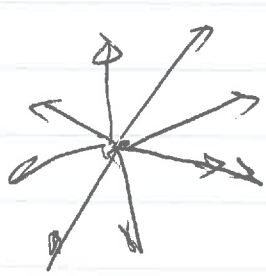
difference $d\vec{E} \parallel \hat{R}$

$d\vec{H} \perp \vec{R}$



For a point charge	For a long wire

Archetypical pictures/models

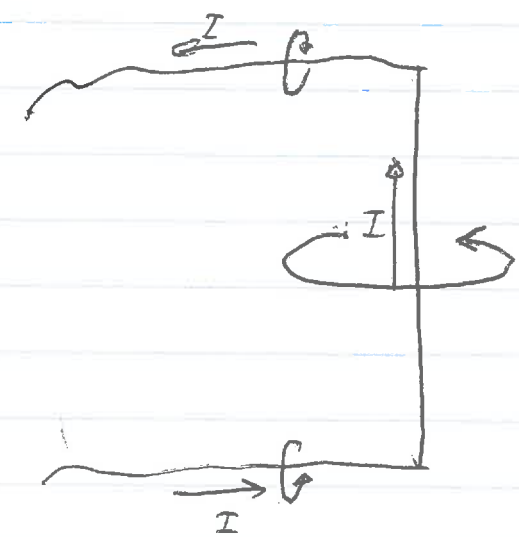


To appreciate the shape of the magnetic field read on your own: Example 5-2, p.p. 244 in 6/E (pp.245 in 7/E)
(i.e. Ex 5-2, p.p. 214 in 5/E).

Magnetic field of a current-carrying wire

Personally I don't like this example too much, coz it's physically not very realistic — the current gotta go somewhere.

The ~~big~~ Good picture is like this.



Take this to the limit, let's say the straight wire is infinitely long then

$$\boxed{\vec{H} = \frac{I}{2\pi r}\,\hat{\phi}}$$

will show how to get this in an easier way later

Before we go further, let's consider the unit of H. It is $\frac{A}{m}$

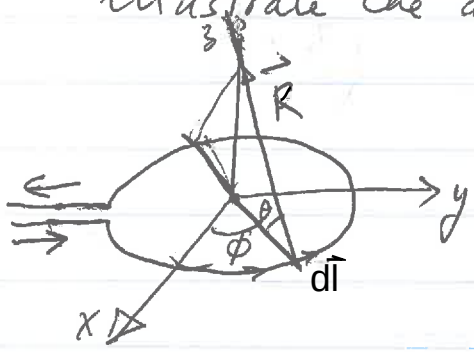
For any loop $\oint dl = ?$

$\oint d\vec{l} = 0$

OK, from that picture we see currents always flow in circuits, which means loops.

last page

So, let's look at a more realistic example to illustrate the application of Biot-Savart law.



Let's say we have a loop like this.

When the gap is small enough, you can ignore it, and the the loop is like the one in the example in the textbook

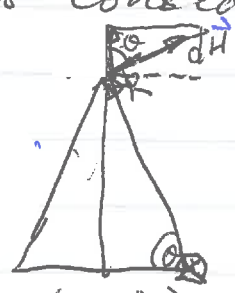
For simplicity, we only attempt to find the H field along the axis of the loop. Call it the z axis.

$$d\vec{H} = \frac{I}{4\pi R^2} d\vec{l} \times \hat{R}$$

Notice that $d\vec{l} \perp \vec{R}$ $\Rightarrow$ $|d\vec{l} \times \hat{R}| = dl$

The horizontal components of the field due to any pair of diametrical elements always cancel each other, so we only need to consider the vertical components.

$$dH_z = \frac{I\,dl}{4\pi R^2} \cos\theta$$



$d\vec{H} \perp \vec{R}$

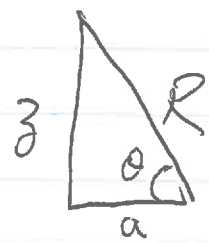
$$H = H_z = \oint dH_z$$

perimeter (circumference)

$$= \oint \frac{I\,dl}{4\pi R^2}\cos\theta = \frac{I\cos\theta}{4\pi R^2}\oint dl$$

$$= \frac{I\cos\theta}{4\pi R^2}\,2\pi a$$

$$= \frac{I}{2R^2}\,a\cos\theta$$



$$= \frac{I\,a}{2(a^2+z^2)}\cdot\frac{a}{\sqrt{a^2+z^2}} = \frac{I a^2}{2(a^2+z^2)^{\frac{3}{2}}}$$

$$\boxed{H(z=0) = \frac{I a^2}{2a^3} = \frac{I}{2a}}$$

$$\boxed{H(z \gg a) = \frac{I a^2}{2|z|^3}}$$

Recall that we defined the magnetic moment

$$m = IA = I\pi a^2$$

$$\therefore\ H(z \gg a) = \frac{m}{2\pi|z|^3}$$

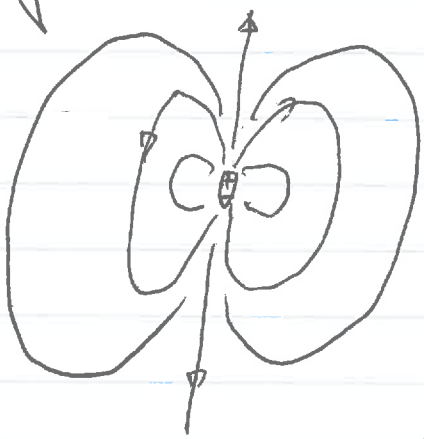
$$\boxed{\vec{H}(z \gg a) = \frac{\vec{m}}{2\pi|z|^3}}$$

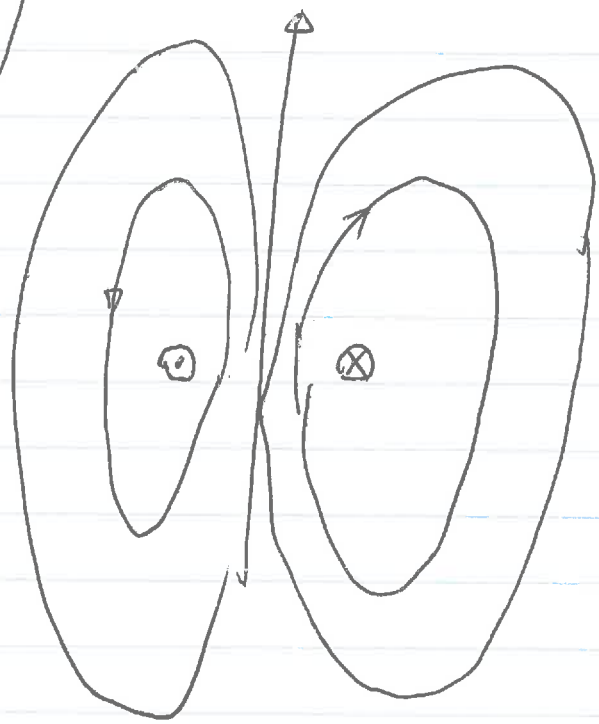
Far from the loop, the shape of the loop doesn't matter.

- Any current carrying loop has a magnetic moment, and the loop is called a magnetic dipole.
- There's no need for a physical wire. For example, an electron orbiting a nucleus forms a magnetic dipole.

We call such a loop a magnetic dipole because

"Its H field has the same shape as the E field of the electric dipole. when we only consider the far field





Here, I stress the far field.

If you zoom in and look at the near field, you'll see a big difference

Have you noticed the difference?

We will comback to this after a necessary digression.

When we studied the electric field, we defined $\vec{D}$ and $\vec{E}$, where $\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$.

$D = \varepsilon_0 E + P$ $\quad$ $\nabla \cdot D = \rho$

Here for the mag field, similarly, we have

$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$. $\quad$ $B = \mu_0 H + M$ $\quad$ $\nabla \times H = J$

μ_r is a property of the material, and μ_0 is a physical constant.

μ_r is dimensionless, which means its just a number without a unit.

There is an important difference between μ_r & ε_r: ε_r has a big range.

$\varepsilon_r \approx 1$ for air, $\quad$ $\varepsilon_r = 2.x$ for many polymers.

$\varepsilon_r = 3.9$ for SiO_2

and for hi-k materials it can be about 10, or 10's. or even hundreds.

There are materials w ε_r in 1000's.

But for μ_r. it's eithe ~ 1, or huge.

Those materials w/ huge μ_r's are called ferromagnetic. such as iron.

They can be permanently magnetized,

which means, when there's no current flowing around there can be a residual B field in them.

For other materials, it's more than sufficient to use $\mu_r = 1$ for most practical purposes.

Now. let's go back and look at an important difference between the magnetic and electric fields.

As we can see from these pictures of the electric dipole and the magnetic dipole, and as I emphasized many times. the electric field line come from the positive charge and end at the negative charge, while the magnetic field lines form loops.

In the mathematical language.

$$\nabla \cdot \vec{D} = \rho \quad \text{or} \quad \oint \vec{D} \cdot d\vec{S} = \int \rho \, dV$$

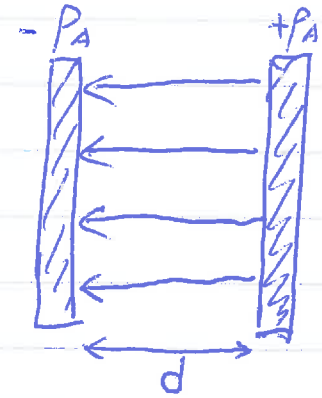
but

$$\nabla \cdot \vec{B} = 0 \quad \text{or} \quad \oint \vec{B} \cdot \underbrace{d\vec{S}}_{\text{surface}} = 0$$

There's no such thing as a "magnetic charge" or monopole.

Let's now give ~~another~~ an example to draw the analogy between the electric & mag fields.

First, for the electric field what's the force per area between the two plates of an infinitely large parallel plate capacitor?



$$E = \frac{\rho_A}{\varepsilon}$$

$$\therefore \frac{F}{A} = \frac{\rho_A A}{A}\,\frac{\rho_A}{\varepsilon} = \frac{\rho_A^2}{\varepsilon}$$

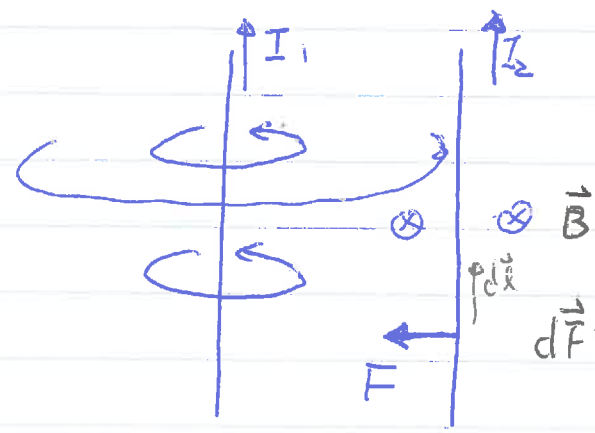
Is this right?

Now, let's look at two wires, carrying currents I_1 & I_2 respectively.
Similarly to the E field situation, here we consider one wire in the field of the other.

The field due to I_1 at the position of wire 2 is

$$H = \frac{I_1}{2\pi d}$$

$$B = \frac{\mu_0 I_1}{2\pi d}$$



$$d\vec{F} = I_2 d\vec{l} \times \vec{B}$$

Of course, the field is a vector, ...

$$\frac{F}{l} = \frac{1}{l} \cdot I_2 l \cdot \frac{\mu_0 I_1}{2\pi d} = \frac{\mu_0 I_1 I_2}{2\pi d}$$ direction.

Notice that two currents flowing in the same direction attract each other

---

Since the dc $\vec{E}$ field must point from the positive charge to the negative charge there can be no loops. Mathematically,

$$\nabla \times \vec{E} = 0 \quad \text{or} \quad \oint \vec{E} \cdot d\vec{l} = 0$$

(conservative)

$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = \rho$

$\oint \vec{D} \cdot d\vec{S} = \varepsilon \oint \vec{E} \cdot d\vec{S} = \int \rho \, dl$

But for the magnetic field, since there's no magnetic monopole, it must form loops

$$\nabla \times \vec{H} = \vec{J} \quad \text{or} \quad \oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} = I$$

$\nabla \cdot \vec{B} = 0$

$\oint \vec{B} \cdot d\vec{S} =$

(solenoidal)

Ampere's law

This is the physics no matter how small the loop is.

Define

$$\text{curl } \vec{H} = \lim_{\Delta S \to 0} \frac{\vec{n} \oint \vec{H} \cdot d\vec{l}}{\Delta S}$$

In the Cartetian coordinates

$$\vec{H} = H_x \hat{x} + H_y \hat{y} + H_z \hat{z}$$

$$\nabla \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(-\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

determinant

Stoke's Theorem — math.

$$\oint_c \vec{H}\cdot d\vec{l}$$

Compare: $\oint \vec{E}\cdot d\vec{l}$

$= \int (\nabla \times \vec{E})\cdot d\vec{S}$ contour c

$= 0$

$$= \sum\Big[\oint_{\substack{\text{small}\\ \text{loop}}} \vec{H}\cdot d\vec{l}\Big]$$

$$= \sum (\nabla\times\vec{H})\cdot\Delta\vec{S} = \int(\nabla\times\vec{H})\cdot d\vec{S} = \int \vec{J}\cdot d\vec{S}$$

physics

Now, go back to the physics:

$$\nabla\times\vec{H} = \vec{J}$$

$$\oint \vec{H}\cdot d\vec{l} = \int \vec{J}\cdot d\vec{S} = I$$

Example.

$$J = \frac{I}{\pi a^2}$$

For $r < a$

$$\oint \vec{H}\cdot d\vec{l} = H(2\pi r) = \frac{I}{\pi a^2}\pi r^2 = I\left(\frac{r}{a}\right)^2$$

$$\therefore H = \frac{1}{2\pi r} I\left(\frac{r}{a}\right)^2 = \frac{Ir}{2\pi a^2}$$

For $r > a$,

$$\oint \vec{H}\cdot d\vec{l} = 2\pi r H = I \qquad \therefore H = \frac{I}{2\pi r}$$

same result as from Biot-Savart law.

For other geometries, read the following examples on your own:

Ex 5-5 (5-6 in 5/E) Toroidal coil.

5-6 (5-7 in 5/E) infinite current sheet

(Think: what about a slab?)

Sect. 5-7.1 (5-8.1 in 5/E) Solenoid

Go thru the mathematical details, but more importantly, get the visual sense of $\vec{B}$ field generated by a current in these different geometries.

Look @ the figures and think.

Make sure they stay in your head.

We will defer the discussion of the inductance to the next chapter, because L is about a changing current. Defining the L in the context of a dc current is not really meaningful.