Dynamic Fields, Maxwell’s Equations (Chapter 6)

So far, we have studied static electric and magnetic fields. In the real world, however, nothing is static. Static fields are only approximations when the fields change very slowly, and “slow” is in a relative sense here.

To really understand electromagnetic fields, we need to study the dynamic fields. You will see the E & M fields are coupled to each other.

Four visual pictures to help you understand the four Maxwell’s equations

Two remain the same for dynamic and static fields. Two are different.

1. \( \int \mathbf{E} \cdot ds = \int \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0} \)

2. \( \int \mathbf{D} \cdot ds = \int \rho dV = Q \)

3. \( \varepsilon \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{D} = \rho \)

This holds for dynamic fields even when \( \rho \) changes with time.

Quiz: how can \( \rho \) change with time?
(2) $\oint \mathbf{B} \cdot ds = 0 \quad \nabla \cdot \mathbf{B} = 0$

What goes in must come out: no monopoles. Always true, static or dynamic.

(3) The electrostatic field is conservative

$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \nabla \times \mathbf{E} = 0$

This is why we can define “potential.”

Faraday’s law:

$\oint \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Pay attention to this negative sign.

This electric field induced by a changing magnetic field is not is not conservative! It’s not an “electrostatic field” even when $\frac{\partial \mathbf{B}}{\partial t}$ is a constant.

Cannot define a potential!
(4) Ampere’s law (static)
\[ \oint H \cdot dl = \int J \cdot dS = I \]
\[ \nabla \times H = J \]

Ampere’s law (dynamic)
\[ \oint H \cdot dl = \int (J + \frac{\partial D}{\partial t}) \cdot dS = I + \int \frac{\partial D}{\partial t} \cdot dS \]
\[ \nabla \times H = J + \frac{\partial D}{\partial t} \]

Displacement current

We have covered the static case in pretty much detail. Here, in the dynamic case the current could include the displacement current.

(3) and (4) are about the coupling between E & M fields. They are the foundations of electromagnetic waves, to be discussed in Ch. 7.
In Ch. 6, we focus on the 3rd eq.

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = -\text{emf} \]

The non-electrostatic, non-conservative field

\[ \frac{\partial \vec{B}}{\partial t} > 0 \]

This "voltage" is called the electromotive force, or emf.

Let's have a digression here, to explain the subtle difference between the emf & a voltage drop. Let's use an analogy of water ducts.

\[ V_{\text{emf}} = \text{emf} \]

\[ V = \text{voltage} \]

\[ V_{\text{emf}} = \oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = -\frac{\partial \Phi}{\partial t} \]

Define \( \Phi = \int \vec{B} \cdot d\vec{s} \) magnetic flux

Unit of \( \Phi \): \( \text{Wb} = T \text{m}^2 \) (weber)

B is therefore called the "magnetic flux density"
We can have a coil of \( N \) turns instead of just one loop.

\[
V = -\frac{d\Phi}{dt}
\]

\[
V_{\text{emf}} = -N \int \frac{\partial \Phi}{\partial t} \, ds
\]

\[
= -N \frac{d\Phi}{dt}
\]

\[
= -\frac{d\Phi}{dt}
\]

\[
\Phi = N \Phi \quad \text{Magnetic flux linkage}.
\]

What's the unit of \( \Phi \)?

Now, let's see what happens if we feed a current to the coil, when there's no external \( B \) field.

\[
\frac{\partial V}{\partial t} > 0
\]

\[
\frac{\partial B}{\partial t} > 0
\]

i will induce a \( B \) field.

This is true, regardless of the shape \# of turns.

But for simplicity, we use the expression for the \( B \) field of a very long solenoid:

\[
B = \mu \frac{N}{L} I
\]

but keep in mind \( B \times I \) in a general case.
If $I$ changes with time, so does $B$:
\[
\frac{dB}{dt} = \mu \left( \frac{N}{l} \right) \frac{di}{dt}, \quad \text{and} \quad \frac{dB}{dt} \propto \frac{di}{dt} \quad \text{in general}
\]

Recall that $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{s}$
\[
V = V_{\text{emf}} = \text{emf} = \oint \vec{E} \cdot d\vec{l} = -N \frac{d}{dt} \int \vec{B} \cdot d\vec{s} = N \frac{d}{dt} \phi = \frac{dB}{dt}
\]

Define $L = \frac{V}{I}$ \implies $\lambda = L i$
\[
\text{then} \quad V = L \frac{di}{dt}
\]

For the coil, or solenoid,
\[
\lambda = N B S = \mu \frac{N^2}{l} i S
\]
\[
\therefore \quad L = \mu \frac{N^2}{l} S
\]

Now I'm going to give you another example of how to find the inductance of a particular geometry of conductors.

For other geometries, read Section 5-7.2 (Fig. 5-27)
Energy stored in an inductor

\[ v = L \frac{di}{dt} \]

\[ w_m = \int iv \, dt = \int i \frac{L}{dt} \, dt = L \int_0^L i \, di = \frac{1}{2} LI^2 \]

Following the math in §5.8 (5.9 in §E)
you get \( w_m = \frac{1}{V} = \frac{1}{2} \mu H^2 \)
for infinitely long solenoid.

Analogy with E field

\[ i = C \frac{dv}{dt} \]
\[ W_E = \frac{1}{2} CV^2 \]
\[ W_E = \frac{1}{2} \varepsilon E^2 \]

\( w_E \) derived from infinitely large cap.

\[ \text{parallel plate} \]
\[ \text{but true for general.} \]

Lastly, the unit

\[ H = \frac{W_b}{A} = \frac{L}{A \cdot m^2} \]
\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} \]

Now we emphasize again the direction of the voltage induced by a changing magnetic field.

Compare this to Fig. 6-2 in the book. (Same # in 5/E)

The voltage current thru an R always flows from hi V to low V.

emf \& V are two different concepts.

This is why I don't like the discussion in the book.

The loop acts like a battery.

In the loop, current flows from low V to hi V. But on the load, the current flows from hi V to low V.

That's why we say a battery or this loop is like a pump.
\[ \vec{B} = (\hat{y} B_{y0} + \hat{z} B_{z0}) \sin \omega t \]

\[ \Phi = \oint \vec{B} \cdot d\vec{s} = B_{z0} \sin \omega t \pi a^2 \]

\[ \Lambda = N \Phi = N B_{z0} \pi a^2 \sin \omega t \]

\[ V = \frac{d\Lambda}{dt} = \omega N B_{z0} \pi a^2 \cos \omega t \]

\[ I = \frac{V}{R} = \frac{\pi N B_{z0} a^2 \omega}{R} \cos \omega t \]

Now, let's see what happens if we put two coils very close to each other. In other words, the \( \vec{B} \) through coil 2 is due to the current through coil 1.

\[ \Phi_{12} = \oint \vec{B}_1 \cdot d\vec{s}_2 \propto I_1 \]

\[ \Lambda_{12} = N_2 \Phi_{12} \equiv L_{12} I_1 \]

\[ V_2 = \frac{d\Lambda_{12}}{dt} = L_{12} \frac{dI_2}{dt} \]

Similarly, \[ V_1 = \frac{d\Lambda_{12}}{dt} = L_{21} \frac{dI_2}{dt} \]

If you do some due diligence mathematically, which we won't do here. Of course, you can show \( L_{12} = L_{21} \). This is called the mutual inductance.

More general discussions...
Now, let's say we have some material, with a very high $\mu$, as the core, and the two coils are around this core.

(Ideal transformer: $\mu = \infty$)

The same $\frac{d\Phi}{dt}$ assuming $\frac{df}{dt} > 0$

Coil 1 must establish a voltage that equals $V_i$.

$$V_i = -N_1 \frac{d\Phi}{dt}$$

How to determine the direction of the $B$ field.

For the primary coil, follow the right-hand rule. For the secondary, opposite to the right hand rule.

**Why: energy conservation.**

Also from energy conservation:

$$V_1 I_1 = V_2 I_2 \implies \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$R_{\text{in}} = \frac{V_1}{I_1} = \frac{N_1 N_2 V_2}{N_1 N_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

For sinusoidal signals.

$$Z_{\text{in}} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

Often used to match impedances.
To help you better understand the magnetic emf, let's see what happens if a piece of conductor is moving through a magnetic field $\vec{F} = \rho \cdot (\vec{v} \times \vec{B})$

$$\text{emf} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$= \nu \vec{B} \cdot l$$

\[V = \nu \vec{B} \cdot l\]

Compare the fig. to Fig 6-8 in the book.

(same # in 5/E)

There's another way to look at this.

$$\Phi = B \cdot l$$

$$\frac{d\Phi}{dt} = B l \frac{dx}{dt} = \nu \vec{B} \cdot l.$$ \[V = \frac{d\Phi}{dt}\]

Read Ex. 6-3 for the non-uniform field case.

(same in 5/E)

You'll see once again, the two methods are consistent with each other.
Another example of induced "voltage"
(i.e. non-electrostatic driving force -- an electromotive force)

\[ V = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \]

\[ V = \frac{\mu_0 I}{2\pi} \ln \frac{r_2}{r_1} \]

Here, I give you the big picture and show you the method. If you want to plug in some #s, go to ex. 6-5 (same # in 5/E) in the book.

Now, you have the knowledge to understand motors & generators.

Let's first look at motors.
To save time, I'm not gonna draw the 3D picture on the board. I'll draw it in the 2D manner. you can look at page 293 (p.p. 266 in 5/E) for the 3D picture.

For the motor, you feed a current to the loop. Let's call it the mag field drives it to rotate.
Recall that we actually showed this in an earlier class.

For positive charge, \( \mathbf{F} \) is in the same direction.

Torque reversed every 180 degrees.

http://resource.rockyview.ab.ca/rvlc/physics30_BU/Unit_B/m4/p30_m4_l03_p4.html
https://www.youtube.com/watch?v=Y-v27GPK8M4
No torque, but coil keeps rotating due to inertia

If current flows in same direction

When electric current passes through a coil in a magnetic field, the magnetic force produces a torque which turns the DC motor.

Electric current supplied externally through a commutator

Magnetic force $F = ILB$ acts perpendicular to both wire and magnetic field

See also:
https://www.youtube.com/watch?v=Y-v27GPK8M4
AC Motor

Rotates at the frequency of the sine wave: “synchronous motor”. Random rotation direction.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html
The mechanical energy input to a generator turns the coil in the magnetic field.

A voltage proportional to the rate of change of the area facing the magnetic field is generated in the coil. This is an example of Faraday's law.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html
Now let's study the generator.
To generate electricity, we need to turn this loop. We spend mechanical energy to get electric energy.

Let's call this angle \( \alpha \)

\[ \alpha = \omega t + \alpha(0) \]

\[ \mathbf{v} = \omega \frac{\mathbf{w}}{2} \]

emf on one side = \( \int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \)

\[ = \omega \frac{\mathbf{w}}{2} \mathbf{B} \sin \alpha \cdot l \]

total emf = \( \omega \mathbf{w} \mathbf{l} \mathbf{B} \sin \alpha = \omega \mathbf{w} \mathbf{l} \mathbf{B} \sin [\omega t + \alpha(0)] \]

\[ = \omega \mathbf{A} \mathbf{B} \sin [\omega t + \alpha(0)] \]

There is a better way to get this result

\[ \Phi = \int \mathbf{B} \cdot d\mathbf{S} = \mathbf{B} \cdot \hat{n} \mathbf{A} = \mathbf{A} \mathbf{B} \cos \alpha \]

\[ = \mathbf{A} \mathbf{B} \cos [\omega t + \alpha(0)] \]

\[ \text{emf} = -\frac{d\Phi}{dt} = \omega \mathbf{A} \mathbf{B} \sin [\omega t + \alpha(0)] \]
\[ \mathbf{M} = \chi_m \mathbf{H} \]

\[ J_M = -\nabla \times \mathbf{M} \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}} = \mu_0 (\mathbf{J} + J_M) = \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{M} = \mu_0 \nabla \times \mathbf{H} + \mu_0 \nabla \times \mathbf{M} = \mu_0 \nabla \times (\mathbf{H} + \mathbf{M}) \]

\[ = \mu_0 (1 + \chi_m) \nabla \times \mathbf{H} \]

\[ \equiv \mu_0 \mu_r \nabla \times \mathbf{H} \equiv \mu \nabla \times \mathbf{H}, \]

where \( \mu \equiv \mu_0 (1 + \chi_m) \equiv \mu_0 \mu_r \)

\[ \mathbf{P} = \chi_e \varepsilon_0 \mathbf{E} \]

\[ \rho_p = -\nabla \cdot \mathbf{P} \]

\[ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho_{\text{total}} = \frac{1}{\varepsilon_0} (\rho + \rho_p) = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P} \]

\[ \varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho \]

\[ \nabla \cdot (\varepsilon_0 + \chi_e \varepsilon_0) \mathbf{E} \equiv \nabla \cdot \mathbf{D} = \rho, \]

where \( \varepsilon \equiv \varepsilon_0 (1 + \chi_e) \equiv \varepsilon_0 \varepsilon_r, \quad \mathbf{D} \equiv \varepsilon_0 \varepsilon_r \mathbf{E} \equiv \varepsilon \mathbf{E} \)
Dielectric polarization always works against the external electric field.

The magnetization $\mathbf{M}$, however, may be parallel or anti-parallel to the external magnetic field $\mathbf{H}$.

- **Paramagnetic**: $\chi_m > 0$, $\mu_r = 1 + \chi_m > 1$
  \[ \mu_r \approx 1 \]
  \[ \mu \approx \mu_0 \]
- **Diamagnetic**: $\chi_m < 0$, $\mu_r = 1 + \chi_m < 1$

**Ferromagnetic**:

- $\mu_r \gg 1$, nonlinearity, hysteresis

The description we give here is phenomenological – no real understanding.

Now that we have tried to give you a qualitative explanation of diamagnetism and paramagnetism, we **must** correct ourselves and say that it is not possible to understand the magnetic effects of materials in any honest way from the point of view of classical physics. Such magnetic effects are a *completely quantum-mechanical phenomenon*.

It is, however, possible to make some **phoney** classical arguments and to get some idea of what is going on.

-- Richar Feynman