Dynamic Fields, Maxwell's Equations (Chapter 6)

So far, we have studied static electric and magnetic fields. In the real world, however, nothing is static. Static fields are only approximations when the fields change very slowly, and "slow" is in a relative sense here.

To really understand electromagnetic fields, we need to study the dynamic fields. You will see the E & M fields are coupled to each other.

Four visual pictures to help you understand the four Maxwell's equations

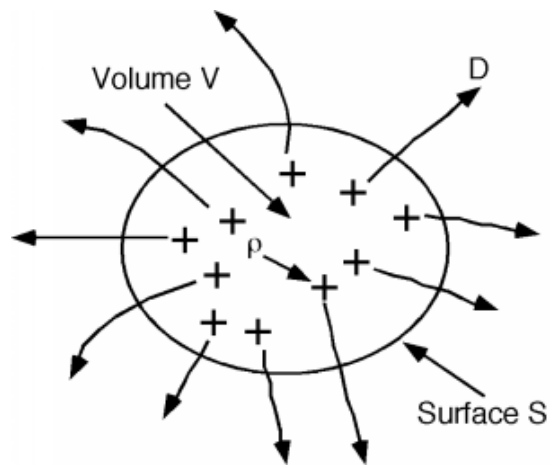
Two remain the same for dynamic and static fields. Two are different.

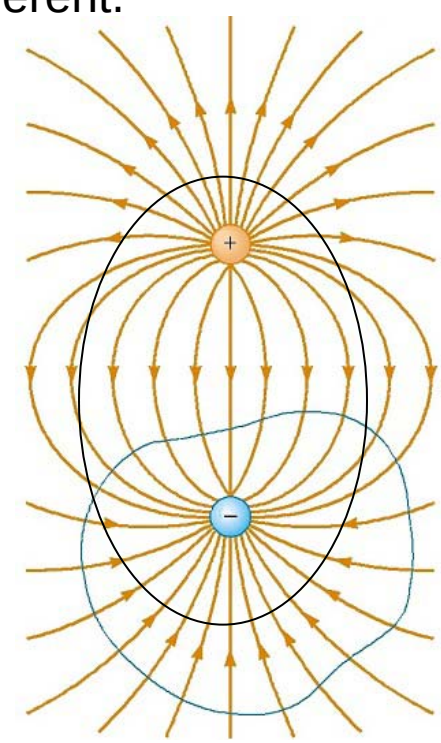
(1)

$$\oint \mathbf{E} \cdot d\mathbf{s} = \int \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

$$\oint \mathbf{D} \cdot d\mathbf{s} = \int \rho dV = Q$$

$$\varepsilon \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{D} = \rho$$

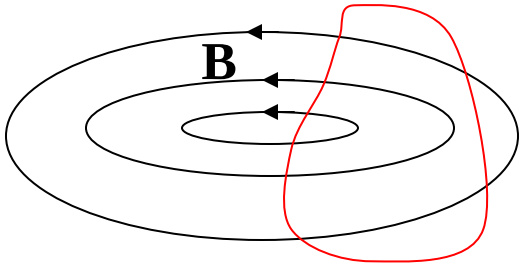




This holds for dynamic fields even when ρ changes with time.

Quiz: how can ρ change with time?

(2) $\oint \mathbf{B} \cdot d\mathbf{s} = 0$ $\nabla \cdot \mathbf{B} = 0$



What goes in must come out: no monopoles.
Always true, static or dynamic.

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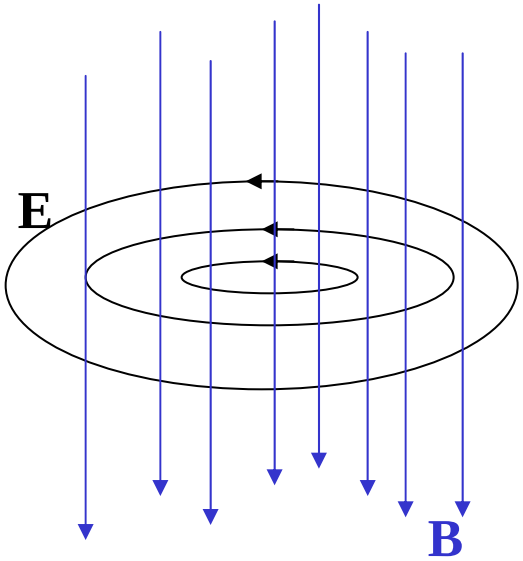
(3) The electrostatic field is conservative

$\oint \mathbf{E} \cdot d\mathbf{l} = 0$ $\nabla \times \mathbf{E} = 0$

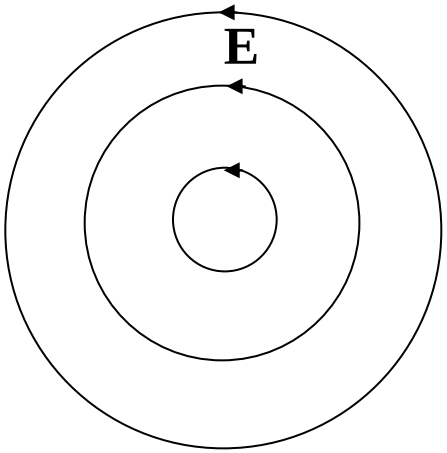
This is why we can define "potential."



Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Pay attention to this negative sign.



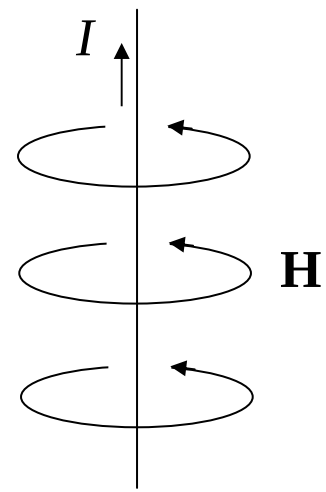
This electric field induced by a changing magnetic field is not is **not conservative**! It's **not** an "electrostatic field" even when $\frac{\partial \mathbf{B}}{\partial t}$ is a constant.

Cannot define a potential!

(4) Ampere's law (static)

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{S} = I$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

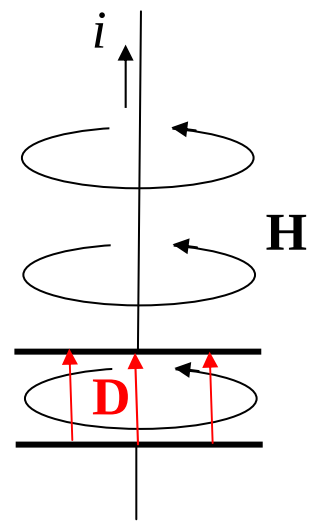


Ampere's law (dynamic)

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S} = I + \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

Displacement current

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$



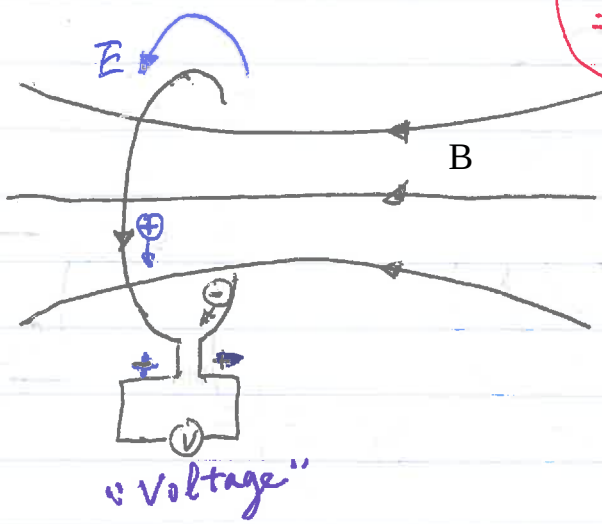
We have covered the static case in pretty much detail. Here, in the dynamic case the current could include the displacement current.

(3) and (4) are about the coupling between E & M fileds. They are the foundations of electromagnetic waves, to be discussed in Ch. 7.

In Ch. 6, we focus on the ~~2nd~~ 3rd eq.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \iff \underbrace{\oint \vec{E}\cdot d\vec{l}}_{=\,\text{emf}} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$

The non-electrostatic, non-conservative field

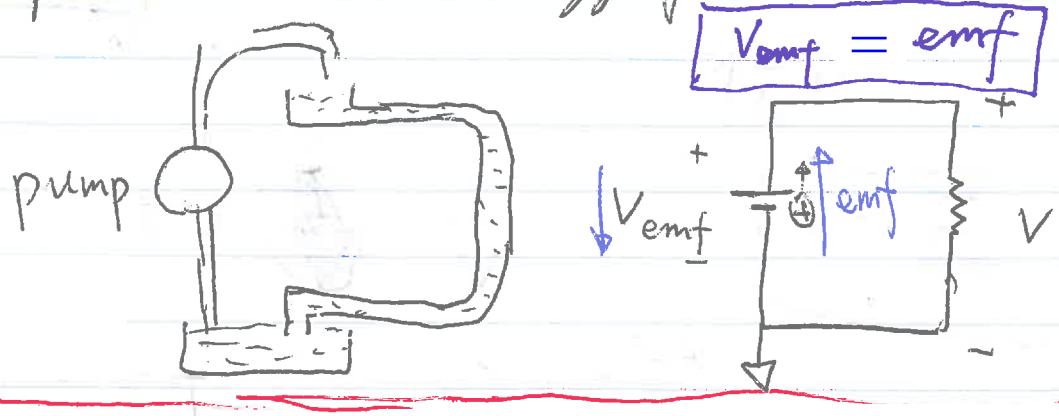


$$\frac{\partial B}{\partial t} > 0$$

This "voltage" is ~~called~~ due to the electromotive force, or emf.

Let's have a digression here, to explain the subtle difference between the emf & a voltage drop. Let's use an analogy of water ducts.

$$\boxed{V_{emf} = \text{emf}}$$



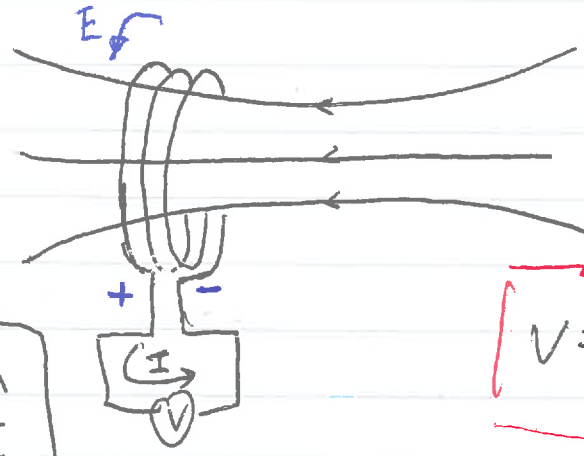
$$V_{emf} = \oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} \equiv -\frac{\partial \Phi}{\partial t} = -\frac{d}{dt}\left(\int \vec{B}\cdot d\vec{S}\right)$$

Define $\Phi = \int \vec{B}\cdot d\vec{S}$ magnetic flux

unit of Φ: Wb $\doteq$ T m^2 (weber)

B is therefore called the "magnetic flux density"

We can have a coil of N turns instead of just one loop



$$V = -N \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$$= -N \frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

$$= -N \frac{d\Phi}{dt}$$

$$= -\frac{d\Lambda}{dt}$$

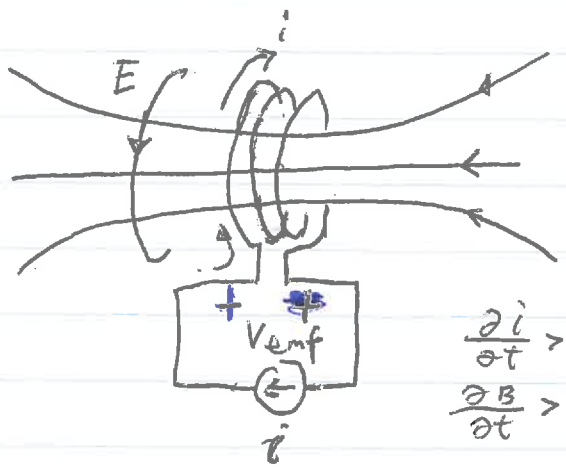
$$\boxed{V = -\frac{d\Lambda}{dt}}$$

$$\boxed{V = \text{emf}}$$

$\Lambda = N\Phi$. Magnetic flux linkage.

What's the unit of Λ ?

Now, let's see what happens if we feed a current to the coil, when there's no external $\vec{B}$ field.



$\frac{\partial i}{\partial t} > 0$

$\frac{\partial B}{\partial t} > 0$.

i will induce a $\vec{B}$ field.

This is true, regardless of the shape or # of turns.

But for simplicity we use the expression for the B field of a very long solenoid

$B = \mu \left(\frac{N}{l}\right) I$ but keep in mind $B \propto I$ in a general case.

If I changes w/ time, so does B

$$\frac{dB}{dt} = \mu\left(\frac{N}{l}\right)\frac{di}{dt}, \quad \text{and} \quad \frac{dB}{dt} \propto \frac{di}{dt} \text{ in general}$$

Recall that $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$

$$v = V_{emf} = \text{emf} = \oint \vec{E}\cdot d\vec{l} = -N\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = -N\frac{d\Phi}{dt} = \frac{d\Lambda}{dt}$$

negative sign due to right hand rule

negative sign gone due to direction of i

Define $L \equiv \frac{\Lambda}{i} \Rightarrow \Lambda = Li$

inductance

then $v = L\frac{di}{dt}$

For the coil or solenoid

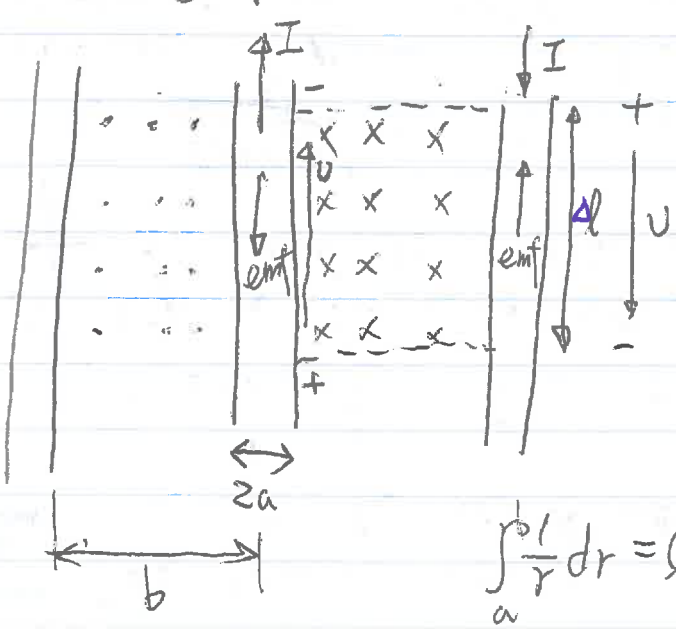
total flux linking the circuit

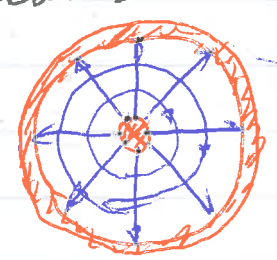
$$\Lambda = NBS = \mu\frac{N^2}{l} i S$$

$$\therefore L = \mu\frac{N^2}{l} S$$

For other geometries, read Section 5-7.2 (Fig. 5-27)

Now I'm going to give you another example of how to find the inductance of a particular geometry of conductors.





$$B = \frac{\mu I}{2\pi r}$$

$$\Lambda = \Phi = l\int_a^b B\,dr$$

$$= \frac{\mu I l}{2\pi}\int_a^b \frac{1}{r}\,dr$$

$$= \frac{\mu I l}{2\pi}\ln\frac{b}{a}$$

$$\int_a^b \frac{1}{r}\,dr = \ln r\Big|_a^b$$

$$\therefore L = \frac{\mu l}{2\pi}\ln\frac{b}{a} \qquad i' = \frac{L}{\Delta l} =$$

(over)

Energy stored in an inductor

$$v = L\frac{di}{dt}$$

$$W_m = \int i v\, dt = \int i\, L\frac{di}{dt}\, dt = L\int_0^I i\, di$$

$$= \frac{1}{2} L I^2$$

Following the math in §5.8 (5.9 in 5/E)

you get $w_m = \frac{W_m}{V} = \frac{1}{2}\mu H^2$

for the infinitely long solenoid.

Analogy w/ E field

Cap	ind
$i = C\frac{dv}{dt}$	$v = L\frac{di}{dt}$
$W_E = \frac{1}{2} C V^2$	$W_m = \frac{1}{2} L I^2$
$w_E = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
derived for the infinitely large cap: parallel plate	.. for the infinitely long solenoid. but ..

archetype

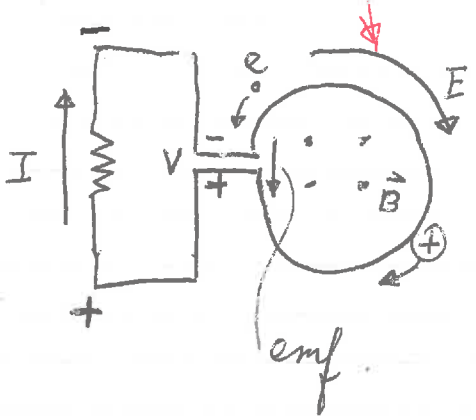
What's the limit here?

but true for general.

Lastly the unit $L = \frac{\Lambda}{I}$

$$H = \frac{Wb}{A} = \frac{T\,m^2}{A}$$

$$\oint \vec{E}\cdot d\ell = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$



$$\frac{\partial B}{\partial t} > 0$$

Now we emphasize again the direction of the voltage induced by a changing mag. field

Compare this to Fig. 6-2 in the book.
(same # in 5/E)

The ~~voltage~~ current thru an R always flows from hi V to low V.

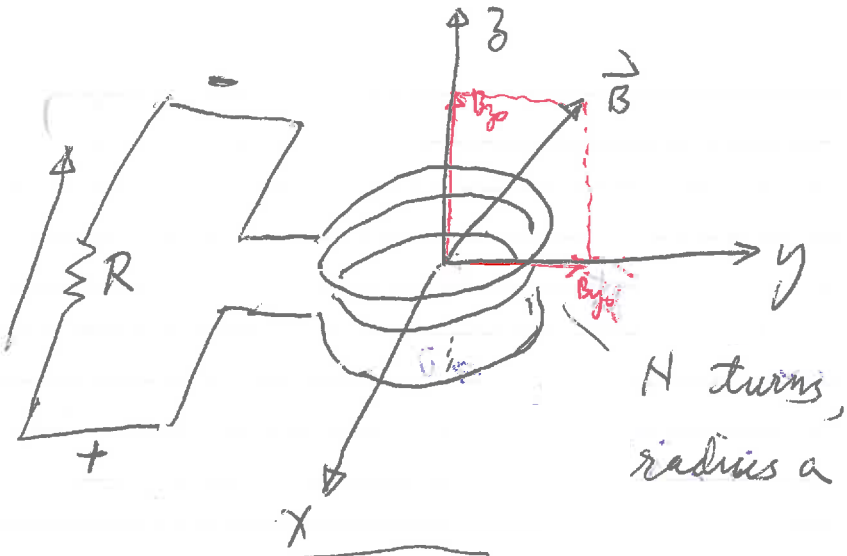
emf & V are two different concepts.

This is why I don't like the ~~[illegible]~~ discussion in the book.

The loop acts like a battery:

In the loop current flows from low V to hi V. but on the load the current flows from hi V to low V.

Thats why we say a battery. or this loop is like a pump.



$$\vec{B} = \left(\hat{y}\, B_{y_0} + \hat{z}\, B_{z_0}\right) \sin\omega t$$

$$\Phi = \int \vec{B} \cdot d\vec{S}$$

$$= B_{z_0} \sin\omega t \, (\pi a^2)$$

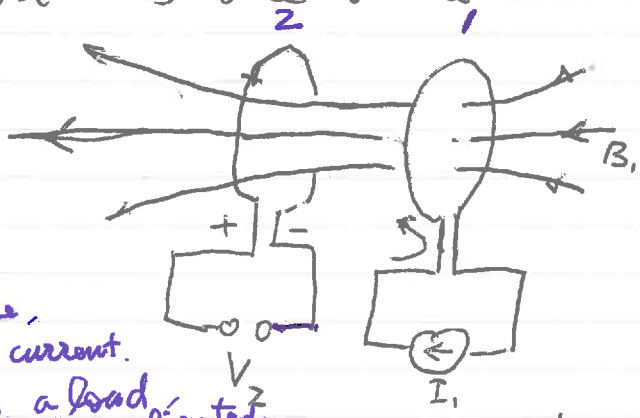
$$\Lambda = N \Phi$$

$$= N B_{z_0} \pi a^2 \sin\omega t$$

$$\boxed{V = \frac{d\Lambda}{dt}} = \omega N B_{z_0} \pi a^2 \cos\omega t$$

$$I = \frac{V}{R} = \frac{\pi N B_{z_0} a^2 \omega}{R} \cos\omega t$$

Now, let's see what happens if we put two coils very close to each other. In other words, the $\vec{B}$ through coil 2 is due to the current ~~[illegible]~~ thru coil 1.



$$\Phi_{12} = \oint_{S_2} \vec{B}_1 \cdot d\vec{S}_2 \propto I_1$$

$$\Lambda_{12} = N_2 \Phi_{12} \equiv L_{12} I_1$$

$$V_2 = \frac{d\Lambda_{12}}{dt} = L_{12} \frac{dI_1}{dt}$$

☆ Here, no current.
If there's a load, more complicated.

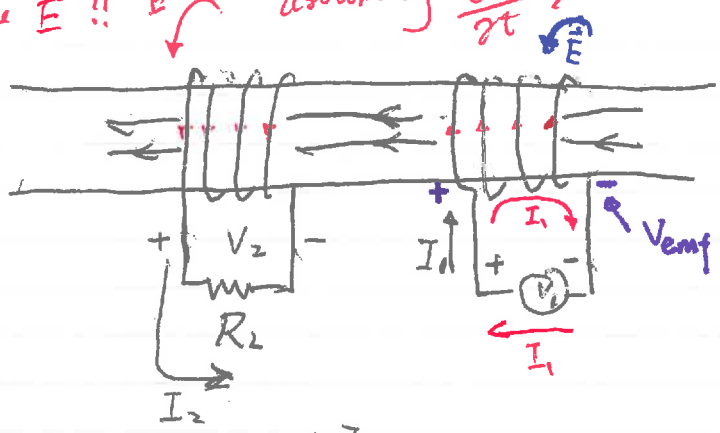
Similarly, $V_1 = \dfrac{d\Lambda_{21}}{dt} = L_{21} \dfrac{dI_2}{dt}$

If you do some "due diligence" mathematically, which we won't do here of course, you can show $L_{12} = L_{21}$. This is called the mutual inductance.

More general discussions ....

Now, let's say we have some material, with a very high μ, as the core, and the two coils are around this core. (Ideal transformer: $\mu = \infty$)

The same $\vec{E}$!! $\vec{E}$ assuming $\frac{\partial I_1}{\partial t} > 0$



$$V_1 = (-)N_1 \frac{d\Phi}{dt}$$

Coil 1 must establish a voltge that equals V_1.

$$V_2 = (-)N_2 \frac{d\Phi}{dt} \qquad \therefore \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

How to determin the direction of the $\vec{B}$ field. For the primary coil, follow the right hand rule. For the secondary, opposite to the right hand rule.

why: energy conservation.

The primary is a load to V_1.
The secondary is a driver.

also from energy conservation.

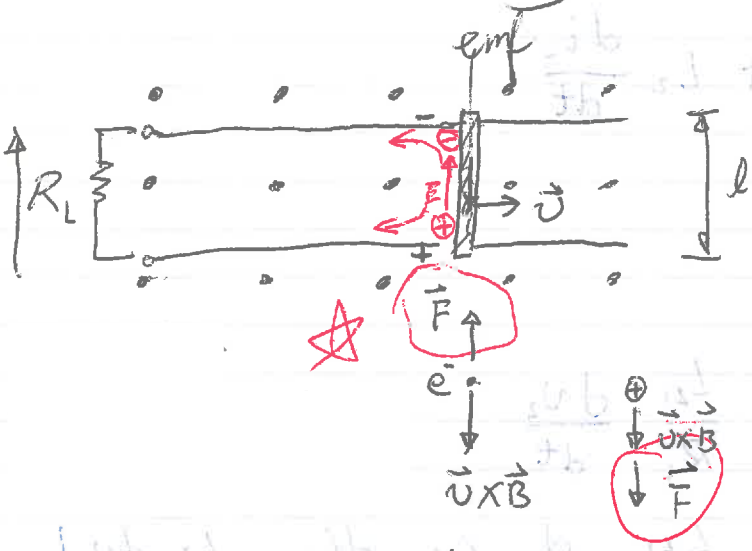
$$V_1 I_1 = V_2 I_2 \Rightarrow \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$R_{in} = \frac{V_1}{I_1} = \frac{\frac{N_1}{N_2} V_2}{\frac{N_2}{N_1} I_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

For sinusoidal signals $Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$

Often used to match impedances.

To help you better understand the magnetic emf. let's see what happens if a piece of conductor is moving thru a mag. field $\vec{F} = q(\vec{v} \times \vec{B})$



$$\boxed{emf = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}}$$
$$= vBl$$

$$V = vBl$$

compare the fig. to Fig 6-8 in the book
(same # in 5/E)

There's another way to look at this.

$$\Phi = B x l$$

B is const., but S is changing.

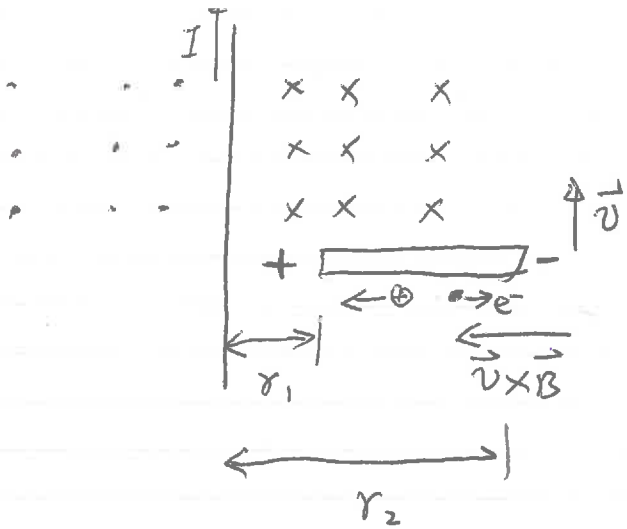
$$\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = vBl \qquad V = \frac{d\Phi}{dt}$$

Read Ex. 6-3 for the non-uniform field case
(same in 5/E)

you'll see once again, the two methods are consistent w/ each other.

Another example of induced "voltage"

(i.e. non-electrostatic driving force -- an electromotive force)



$$V = \int (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= v\int_{r_1}^{r_2} \frac{\mu_0 I}{2\pi r}\, dr$$

$$= \frac{\mu_0 I v}{2\pi}\int_{r_1}^{r_2} \frac{1}{r}\, dr$$

$$= \frac{\mu_0 I v}{2\pi}\ln\frac{r_2}{r_1}$$

Here, I give you the big picture and show you the method. If you want to plug in some #s, go to ex. 6-5 (same # in 5/E) in the book.

Now, you have the knowlege to understand motors & generators.

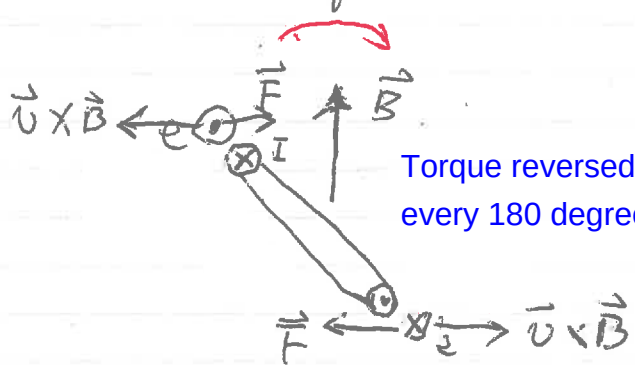
Let's first look @ motors. (dc & ac)

To save time, I'm not gonna draw the 3D picture on the board. I'll draw it in the 2D manner. you can look @ page 293 (p.p. 266 in 5/E) for the 3D picture.

Fig. 6-11

(in 6/E, or pp. 294 in 7/E)

For the motor, you feed a current to the loop. Lets see how the mag. field drives it to rotate.



Torque reversed every 180 degrees.

Recall that we actually showed this in an earlier class.

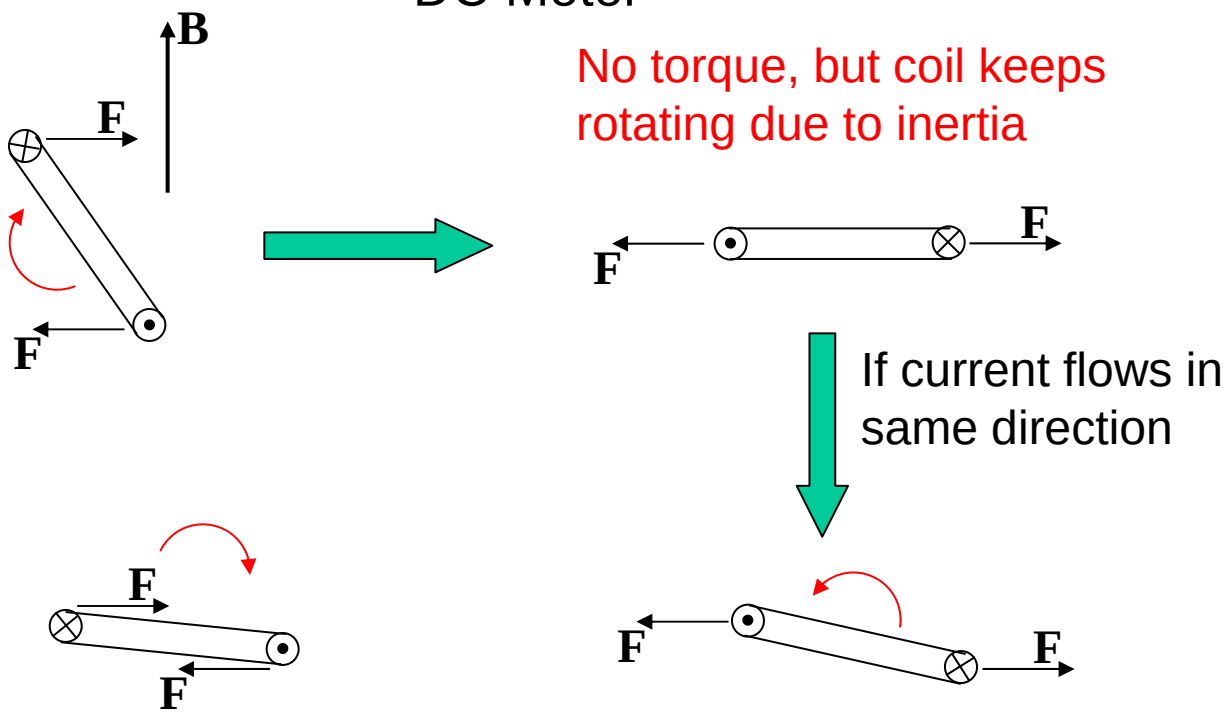
(for the dc motor)

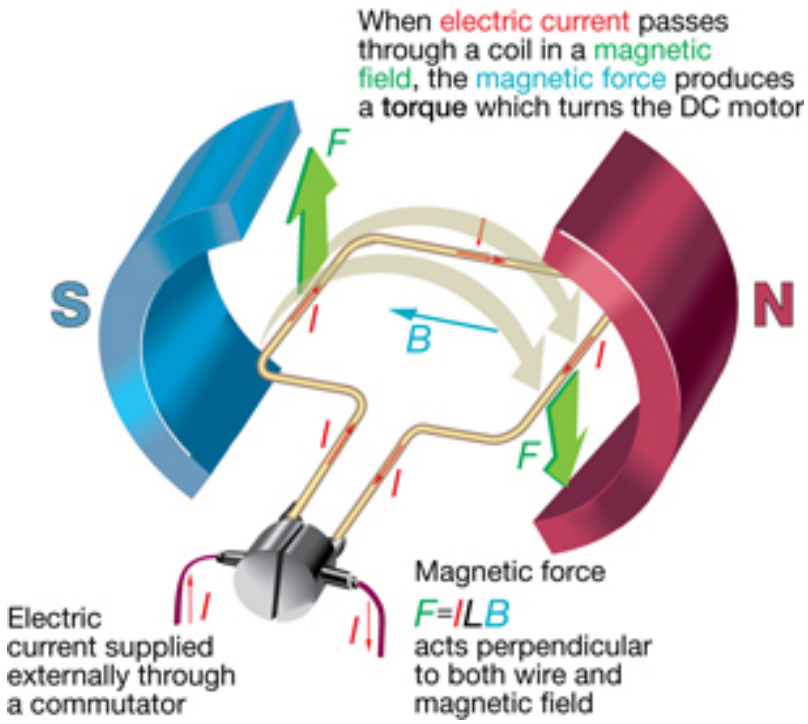
http://resource.rockyview.ab.ca/rvlc/physics30_BU/Unit_B/m4/p30_m4_l03_p4.html

https://www.youtube.com/watch?v=Y-v27GPK8M4

For positive charge $\vec{F}$ is in the same direction

DC Motor

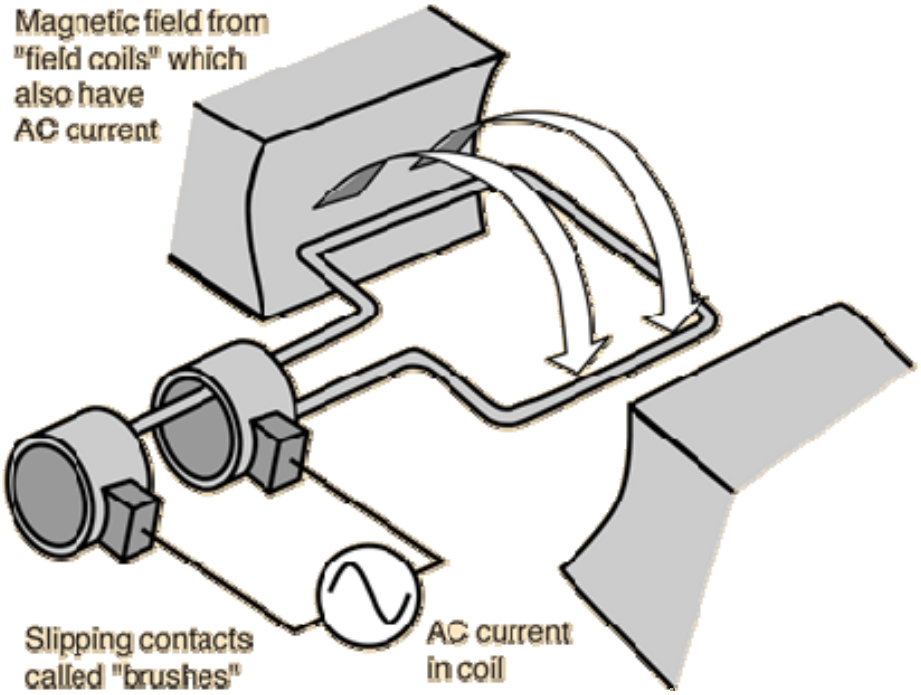




http://resource.rockyview.ab.ca/rvlc/physics30_BU/Unit_B/m4/p30_m4_l03_p4.html

See also:
https://www.youtube.com/watch?v=Y-v27GPK8M4

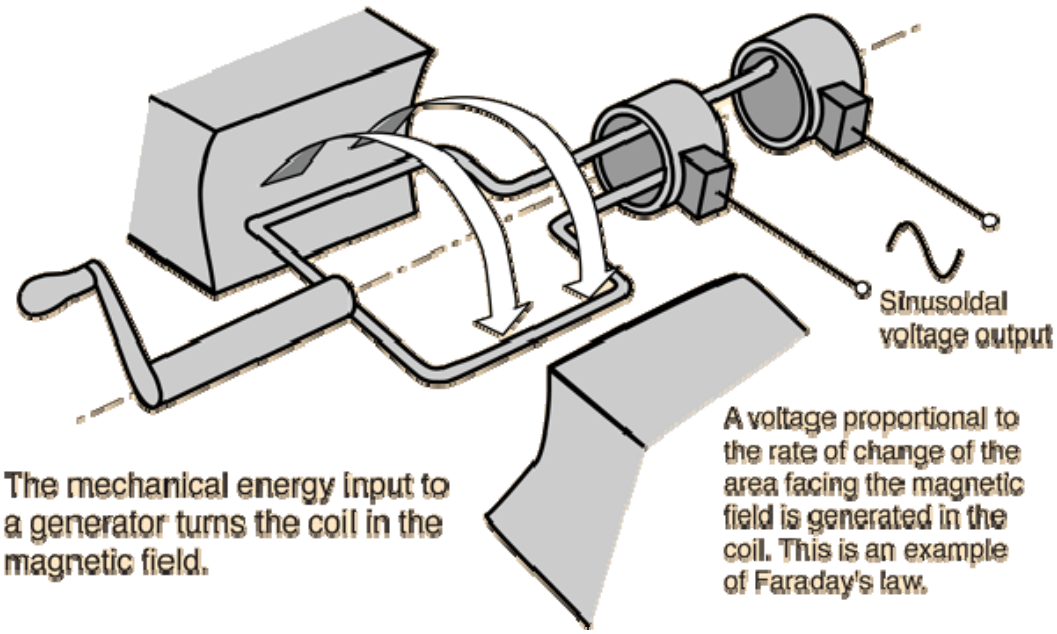
AC Motor



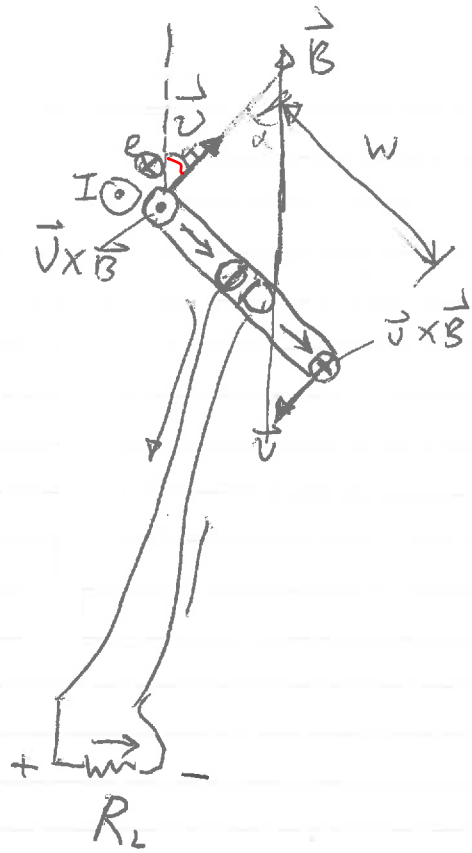
Rotates at the frequency of the sine wave: “synchronous motor”.
Random rotation direction.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html

Generator



http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html



Now let's study the generator.
To generate electricity we need to turn this loop.
We spend mechanic energy to get electric energy.

Let's call this angle α

$$\alpha = \omega t + \alpha(0)$$

$$v = \omega \frac{w}{2}$$

$$\text{emf on one side} = \int (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= \omega \frac{w}{2} B \sin\alpha \cdot l$$

$$\text{total emf} = \omega w l B \sin\alpha = \omega w l B \sin[\omega t + \alpha(0)]$$

$$= \omega A B \sin[\omega t + \alpha(0)]$$

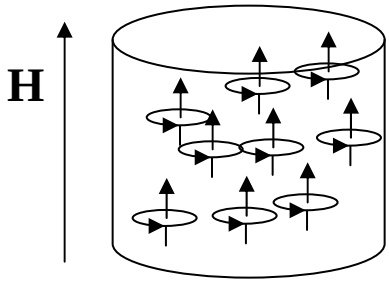
There is a better way to get this result

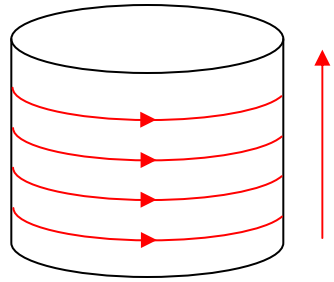
$$\Phi = \int \vec{B}\cdot d\vec{S} = \vec{B}\cdot\hat{n}\, A = AB\cos\alpha$$

$$= AB\cos[\omega t + \alpha(0)]$$

$$\text{emf} = -\frac{d\Phi}{dt} = \omega A B \sin[\omega t + \alpha(0)]$$

Magnetism of Materials





$$\mathbf{M} = \lim_{\Delta V \to 0} \frac{\sum_i \mathbf{m}_i}{\Delta V}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad J_M = -\nabla \times \mathbf{M}$$

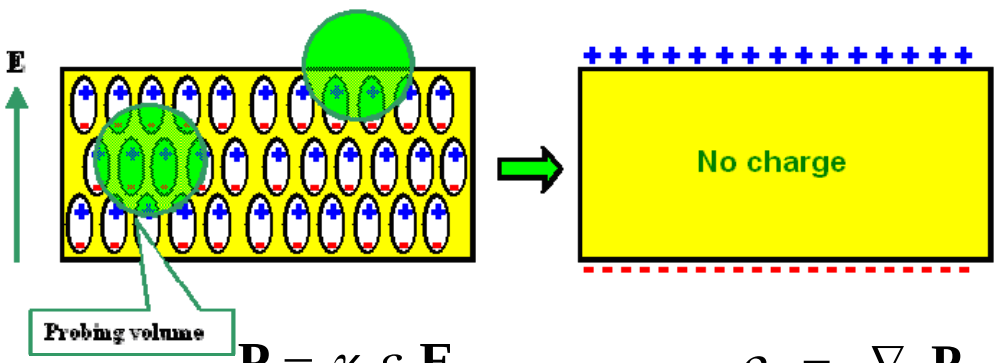
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}} = \mu_0 (\mathbf{J} + \mathbf{J}_M) = \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{M}$$

$$= \mu_0 \nabla \times \mathbf{H} + \mu_0 \nabla \times \mathbf{M} = \mu_0 \nabla \times (\mathbf{H} + \mathbf{M})$$

$$= \mu_0 (1 + \chi_m) \nabla \times \mathbf{H}$$

$$\equiv \mu_0 \mu_r \nabla \times \mathbf{H} \equiv \mu \nabla \times \mathbf{H},$$

where $\mu \equiv \mu_0 (1 + \chi_m) \equiv \mu_0 \mu_r$



$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{\sum_i \mathbf{p}_i}{\Delta V}$$

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E} \qquad \rho_P = -\nabla \cdot \mathbf{P}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho_{\text{total}} = \frac{1}{\varepsilon_0} (\rho + \rho_P) = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P}$$

$$\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho$$

$$\nabla \cdot (\varepsilon_0 + \chi_e \varepsilon_0) \mathbf{E} \equiv \nabla \cdot \mathbf{D} = \rho \quad ,$$

where $\varepsilon \equiv \varepsilon_0 (1 + \chi_e) \equiv \varepsilon_0 \varepsilon_r$, $\mathbf{D} \equiv \varepsilon_0 \varepsilon_r \mathbf{E} \equiv \varepsilon \mathbf{E}$

Dielectric polarization always works against the external electric field.

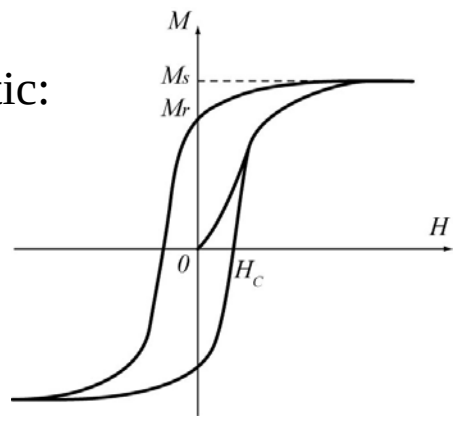
The magnetization **M**, however, may be parallel or anti-parallel to the external magnetic field **H**.

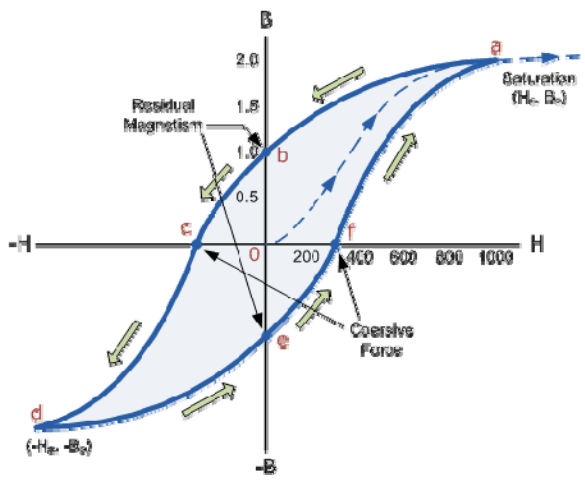
Paramagnetic: $\chi_m > 0$, $\mu_r = 1 + \chi_m > 1$

Diamagnetic: $\chi_m < 0$, $\mu_r = 1 + \chi_m < 1$

$$\boxed{\begin{array}{l}\mu_r \approx 1 \\ \mu \approx \mu_0\end{array}}$$

Ferromagnetic:
$\mu_r >> 1$,
nonlinearity,
hysteresis





The description we give here is phenomenological – no real understanding.

Now that we have tried to give you a qualitative explanation of diamagnetism and paramagnetism, we ***must*** correct ourselves and say that *it is not possible* to understand the magnetic effects of materials in any honest way from the point of view of classical physics. Such magnetic effects are a *completely quantum-mechanical phenomenon*.

It is, however, possible to make some ***phoney*** classical arguments and to get some idea of what is going on.

-- Richar Feynman