Phasors

Handling the trigonometric functions can be tedious:

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

This is just a simple phase shift, say, \( \alpha = \omega t, \beta = \varphi_0 \).

We have Euler's identity: \( e^{ix} = \cos x + i \sin x \)

that relates trigonometric functions to complex exponentials,

the math of which is a bit simpler:

\[ e^{j(\alpha + \beta)} = e^{j\alpha} e^{j\beta} \]

At least, phase shifting is much easier!

So, we use \( e^{j\alpha} \) to represent \( \cos \alpha \) for mathematical convenience.

But, why can we do this? What’s behind this?
\[ e^{j \omega t} = \cos \omega t + j \sin \omega t \]

\[ \theta = \omega t + \phi \]

The projections of circular motion are harmonic oscillations.

\[ \text{Re } e^{j \omega t} = \cos \omega t \]
\[ \text{Im } e^{j \omega t} = \sin \omega t \]

For every oscillation \( A \cos(\omega t + \phi) \), we add an "imaginary" partner \( jA \sin(\omega t + \phi) \)

\[ \Rightarrow \quad \text{A } e^{j(\omega t + \phi)} = A \cos(\omega t + \phi) + jA \sin(\omega t + \phi) \]

In stead of considering \( A \cos(\omega t + \phi) \), which is mathematically more complicated, we consider \( A e^{j(\omega t + \phi)} \), which is mathematically simpler.

Why can we do this?

— Linearity!

Both \( \cos \omega t \) & \( \sin \omega t \) are solutions of the Wave equation, so is their linear combo.
Based on this, we have a math tool to handle harmonic osc. more easily.

It can be a pain to handle cos & sin functions so we add an "imaginary" partner to a cos function

\[ \cos wt = e^{jwt} = \cos wt + j\sin wt \]

\[ v(t) = A \cos(wt+\phi) \rightarrow A e^{j(wt+\phi)} \]

(say, a voltage)

This rotation part is always there. So leave it out.

(or "complex amplitude"

\[ \text{phasor } \vec{V} = Ae^{j\phi_0} = A \angle \phi_0 \]

-- same as you have learned in circuit theory and signals and systems.

For a wave,

\[ v(x,t) = A \cos(wt - \beta x + \phi_0) \]

\[ \rightarrow Ae^{j(-\beta x + \phi_0)}e^{jwt} \]

leave out.

\[ \text{phasor } \vec{V}(x) = Ae^{-j(\beta x + \phi_0)} = A \angle (\beta x + \phi_0) \]

\[ = (Ae^{j\phi_0})e^{-j\beta x} \]

\[ \rightarrow x \text{ dependence.} \]

In both cases, the time dependence is left out, since we are dealing with a single frequency.
Advantages — an example

\[ u(t) = V_0 \cos \omega t \]

Find \( i(t) \).

(This is a special case of the example of Fig. 1-20 in the textbook with \( \phi_0 = \frac{\pi}{2} \))

The \( \phi_0 \) defined there, in a sine reference.

\[ \tilde{V} = V_0 \]

\[ v(t) = \text{Re} (\tilde{V} e^{j\omega t}) = \text{Re} (V_0 e^{j\omega t}) \]

\[ = V_0 \cos \omega t \]

Try to solve \( i(t) \) in the time domain:

\[ \frac{1}{C} \int i(t) \, dt + R i(t) = V_0 \cos \omega t \]

\[ \frac{1}{C} \cdot i(t) + R \frac{di}{dt} = -\omega V_0 \sin \omega t \]

— need to solve the differential equation.
A good thing about complex exponentials:
\[
\frac{d}{dt} e^{jwt} = jwe^{jwt}, \quad \int e^{jwt} dt = \frac{l}{jw} e^{jwt}
\]
Differential equations are turned into algebra.
\[
\frac{1}{C} \int i(t) dt + Ri(t) = V_0 \cos jwt
\]
Adding "imaginary partners" to both \(i(t)\) & \(v(t)\):
\[
\frac{1}{C} \int \tilde{I} e^{jwt} dt + R \tilde{I} e^{jwt} = V_0 e^{jwt}
\]
Notice that \(\tilde{I}\) is a "constant" — no \(t\) dependence
\[
\frac{1}{jwc} \tilde{I} e^{jwt} + R \tilde{I} e^{jwt} = V_0 e^{jwt} = \tilde{V} e^{jwt}
\]
\[
\tilde{I} (\tilde{R} + \frac{l}{jwc}) = V_0 = \tilde{V} \quad \Rightarrow \quad \tilde{I} = \frac{V_0}{\tilde{R} + \frac{l}{jwc}}
\]
Now, relate the phasor back to the time domain:
\[
\frac{1}{R + \frac{l}{jwc}} = \frac{jwc}{1 + jwRC} = \frac{wCe^{j\frac{\pi}{2}}}{\sqrt{1 + w^2R^2C^2}} e^{j\phi_1}
\]
\[
= \frac{wC}{\sqrt{1 + w^2R^2C^2}} e^{j(\frac{\pi}{2} - \phi_1)}
\]
where \(\tan \phi_1 = wRC\), i.e. \(\phi_1 = \tan^{-1}(wRC) = \arctan(wRC)\)
\[ I = \frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j\left(\frac{\pi}{2} - \phi_1\right)} \]

\[ i(t) = \text{Re} \left( I e^{j\omega t} \right) \]

\[ = \text{Re} \left[ \frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j\left(\omega t + \frac{\pi}{2} - \phi_1\right)} \right] \]

\[ = \frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}} \cos \left(\omega t + \frac{\pi}{2} - \phi_1\right) \]

*Amplitude* \quad *Phase*

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**Important:**

Instantaneous value $\leftrightarrow$ phasor

*Table 1-5 (in both 7/E & 6/E of the book)*