Phasors

Handling the trigonometric functions can be tedious:

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

This is just a simple phase shift, say, $\alpha = \omega t$, $\beta = \varphi_0$.

We have Euler's identity: $e^{ix} = \cos x + i \sin x$

that relates trigonometric functions to complex exponentials,

the math of which is a bit simpler:

$$e^{j(\alpha+\beta)} = e^{j\alpha} e^{j\beta}$$

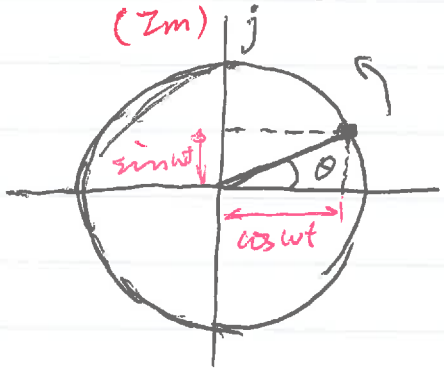
At least, phase shifting is much easier!

So, we use $e^{j\alpha}$ to represent $\cos\alpha$ for mathematical convenience.

But, why can we do this? What's behind this?

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

The projections of circular motion are harmonic oscillations.



$$\theta = \omega t + \phi$$

$$\text{Re}\, e^{j\omega t} = \cos\omega t$$
$$\text{Im}\, e^{j\omega t} = \sin\omega t$$

For every oscilation $A\cos(\omega t + \phi)$, we add an "imaginary" partner $jA\sin(\omega t + \phi)$

$\Rightarrow$

$$Ae^{j(\omega t+\phi)}$$
$$= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$$

In stead of considering $A\cos(\omega t + \phi)$, which is mathematically more complicated, we consider $Ae^{j(\omega t+\phi)}$, which is mathematically simpler.

Why can we do this?

— Linearity!

Both $\cos\omega t$ & $\sin\omega t$ are solutions of the Wave equation, so is their linear combo.

Based on this, we have a math tool to handle harmonic osc. more easily.

It can be a pain to handle cos & sin functions so, we add an "imaginary" partner to a cos function

$$\cos\omega t \rightarrow e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$v(t) = A\cos(\omega t+\phi_0) \rightarrow A e^{j(\omega t+\phi_0)} \quad \text{(say, a voltage)}$$

$$= \left(A e^{j\phi_0}\right) e^{j\omega t}$$

This rotation part is always there.

So leave it out.

$$\text{phasor } \tilde{V} = A e^{j\phi_0} = A\angle\phi_0$$

(or "complex amplitude")

— Same as you have learned in circuit theory and signals and systems.

For a wave,

$$v(x,t) = A\cos(\omega t - \beta x + \phi_0)$$

$$\rightarrow A e^{j(-\beta x+\phi_0)} e^{j\omega t}$$

Leave out.

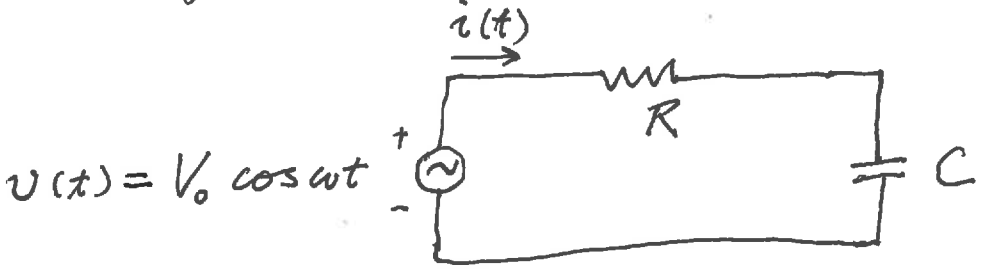
$$\text{Phasor } \tilde{V}(x) = A e^{-j(\beta x+\phi_0)} = A\angle(-\beta x+\phi_0)$$

$$= \left(A e^{j\phi_0}\right) e^{-j\beta x}$$

$\rightarrow$ x dependence.

In both cases, the time dependence is left out, since we are dealing with a single frequency.

Advantages — an example



Find $i(t)$.

(This is a special case of the example of Fig. 1-20 in the textbook with $\phi_0 = \frac{\pi}{2}$)

↳ the ϕ_0 defined there, in a sine reference.

$$\tilde{V} = V_0$$

$$v(t) = \mathrm{Re}\left(\tilde{V} e^{j\omega t}\right) = \mathrm{Re}\left(V_0 e^{j\omega t}\right)$$

$$= V_0 \cos \omega t$$

Try to solve $i(t)$ in the time domain:

$$\frac{1}{C}\int i(t)\,dt + R\,i(t) = V_0 \cos \omega t$$

$$\frac{1}{C}\, i(t) + R \frac{di}{dt} = -\omega V_0 \sin \omega t$$

—— need to solve the differential equation.

A good thing about complex exponentials:

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}, \quad \int e^{j\omega t}\,dt = \frac{1}{j\omega}e^{j\omega t}$$

Differential equations are turned into algebra.

$$\frac{1}{C}\int i(t)\,dt + Ri(t) = V_0\cos\omega t$$

Adding "imaginary partners" to both $i(t)$ & $v(t)$:

$$\frac{1}{C}\int \tilde{I}e^{j\omega t}\,dt + R\tilde{I}e^{j\omega t} = V_0 e^{j\omega t}$$

Notice that $\tilde{I}$ is a "constant" — no t dependence

$$\frac{1}{j\omega C}\tilde{I}e^{j\omega t} + R\tilde{I}e^{j\omega t} = V_0 e^{j\omega t} = \tilde{V}e^{j\omega t}$$

$$\boxed{\tilde{I}\left(R + \frac{1}{j\omega C}\right) = V_0 = \tilde{V}} \quad \Rightarrow \tilde{I} = \frac{V_0}{R + \frac{1}{j\omega C}}$$

Now, relate the phasor back to the time domain:

$$\frac{1}{R + \frac{1}{j\omega C}} = \frac{j\omega C}{1 + j\omega RC} = \frac{\omega C e^{j\frac{\pi}{2}}}{\sqrt{1+\omega^2R^2C^2}\,e^{j\phi_1}}$$

$$= \frac{\omega C}{\sqrt{1+\omega^2R^2C^2}}e^{j\left(\frac{\pi}{2}-\phi_1\right)},$$

where $\tan\phi_1 = \omega RC$, i.e. $\phi_1 = \tan^{-1}(\omega RC) = \arctan(\omega RC)$

$$\therefore \quad \tilde{I} = \frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} e^{j\left(\frac{\pi}{2} - \phi_1\right)}$$

$$i(t) = \mathrm{Re}\left(\tilde{I} e^{j\omega t}\right)$$

Put the $e^{j\omega t}$ back & take the real part

$$= \mathrm{Re}\left[\frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} e^{j\left(\omega t + \frac{\pi}{2} - \phi_1\right)}\right]$$

$$= \frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} \cos\left(\omega t + \frac{\pi}{2} - \phi_1\right)$$

Amplitude — phase

Important:

Instantaneous value $\Longleftrightarrow$ phasor

Table 1-5 (in both 7/E & 6/E of the book)