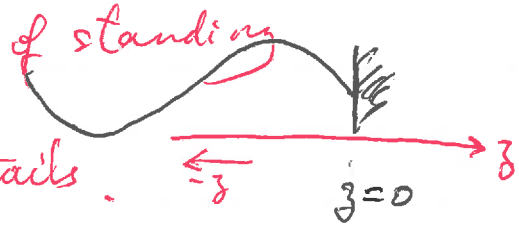
we talked about the qualitative picture of standing waves. Here are the quantitative details.



Displacement of a string with one end fixed

$$\tilde{Y} = Y_0^+ e^{-j\beta z} + Y_0^- e^{-\beta z}$$

Phasor representation. You should know how to convert to time domain.

$$\tilde{Y}(0) = 0 \Rightarrow Y_0^+ + Y_0^- = 0$$

$$Y_0^- = -Y_0^+ \qquad \boxed{\Gamma = -1}$$

(There was a homework problem)

$$\tilde{Y} = Y_0^+ \left(e^{-j\beta z} - e^{j\beta z}\right)$$

$$\boxed{e^{j\theta} - e^{-j\theta} = 2j\sin\theta}$$

$$\tilde{Y} = -2Y_0^+ j\sin(\beta z)$$

The phasor is actually the complex amplitude.

For simplicity, assume Y_0^+ is positive and real.

$$y(z,t) = \mathrm{Re}\left[-2jY_0^+ \sin(\beta z)\, e^{j\omega t}\right]$$

$$\Rightarrow 2Y_0^+ \sin(\beta z) \sin\omega t$$

Quiz:

What property of a mirror makes it a mirror?

at the other end. $Z_g = Z_0$. equivalent to semi-infinitely long

similarly

$\Gamma = -1$, $V_0^- = -V_0^+$

$Z_L = 0$ S.C.

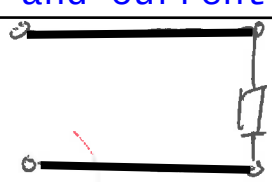
$$\tilde{V}(z) = V_0^+ \left(e^{-j\beta z} - e^{j\beta z}\right) = -2V_0^+ j\sin\beta z$$

But, $I_0^+ = I_0^-$ $\left(\frac{I_0^-}{I_0^+} = -\Gamma = +1\right)$

What's the current? Convert voltage and current phasors to time domain.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

$$\therefore V_0^+ = V_0^-$$



$Z_L = \infty$ O.C

In practice, not easy to make an open circuit.

$$I_0^+ = -I_0^- \Rightarrow I_L = 0$$

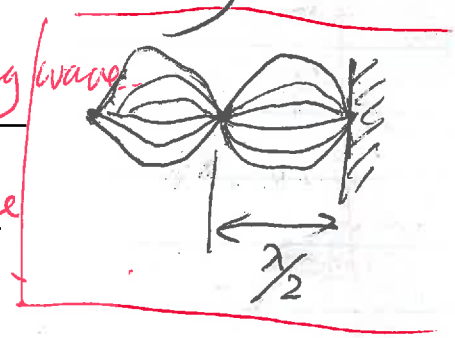
$$\tilde{V}(z) = V_0^+ \left(e^{-j\beta z} + e^{j\beta z}\right)$$

$$= 2V_0^+ \cos \beta z$$

Again, find the current. Convert both voltage and current phasors to time domian.

When $\Gamma = \pm 1$, completely standing wave.

Refer to Wiki ped. Standing wave



~~For example,~~ At very high frequencies, we often can only measure the amplitude.

For short circuit,

The amplitude of $\tilde{V}$ is

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\,\tilde{V}^*(z)}$$

$$= |-2V_0^+ \sin \beta z|$$

$$= 2|V_0^+|\,|\sin \beta z|$$

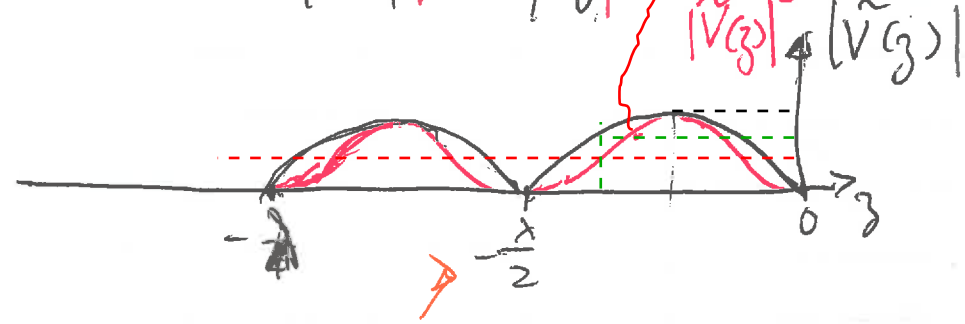
$$|V(z)|^2 = 4|V_0^+|^2 \sin^2 \beta z$$

$$= 2(1 - \cos 2\beta z)$$

Quiz:

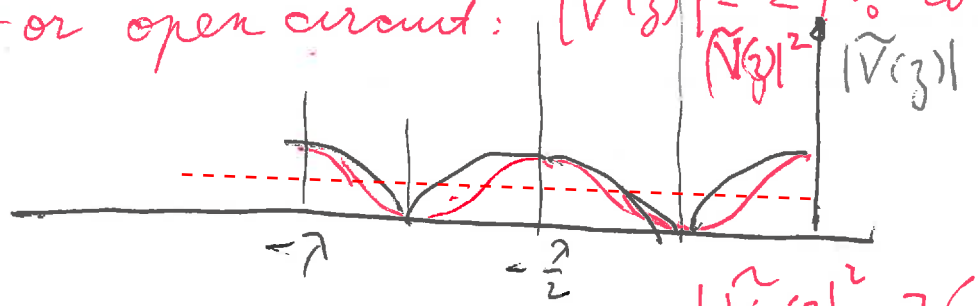
What's the spatial period of the standing wave?

$\Gamma = -1$



for short ckt

For open circuit: $|\tilde{V}(z)| = 2|V_0^+ \cos \beta z|$



$$|\tilde{V}(z)|^2 = 2(1 + \cos 2\beta z)$$

O.C.

explain

$$|\tilde{V}|_{max} = 2|V_0^+|$$

$$|\tilde{V}|_{min} = 0$$

— This is said to be completely a standing wave. $|\Gamma| = 1$.

What if $|\Gamma| \neq 1$?

partially standing, partially traveling.

Now, let's look @ the max & min of this combination of standing wave & traveling wave.

Recall that, in general Γ is a complex

$$\Gamma = |\Gamma| e^{j\theta_r}$$

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\ \tilde{V}^*(z)}$$

$$= \sqrt{V_0^+(e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z})\ (V_0^+)^*(e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z})}$$

V_0^+ is a complex amplitude

notice that $V_0^+ (V_0^+)^* = |V_0^+|^2$

$$\therefore |\tilde{V}(z)| = |V_0^+| \sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+| \sqrt{1 + |\Gamma|^2 + \underbrace{2|\Gamma| \cos(2\beta z + \theta_r)}_{\text{interference term}}}$$

① Lin-f only

We can measure $|V|^2$

similarly,

★ Resistive $\Rightarrow$ $\theta_r = 0$ or π. max or min

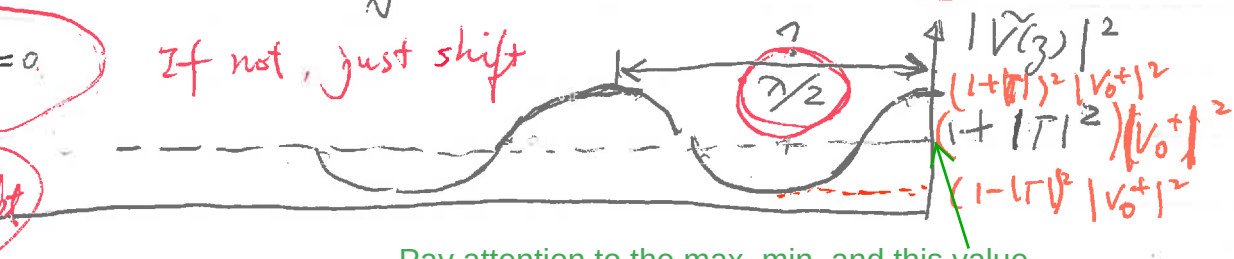
$$|\tilde{I}(z)| = |I_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta z + \theta_r)}$$

If $\theta_r = 0$

open ckt

$Z_L > Z_0$

If not, just shift

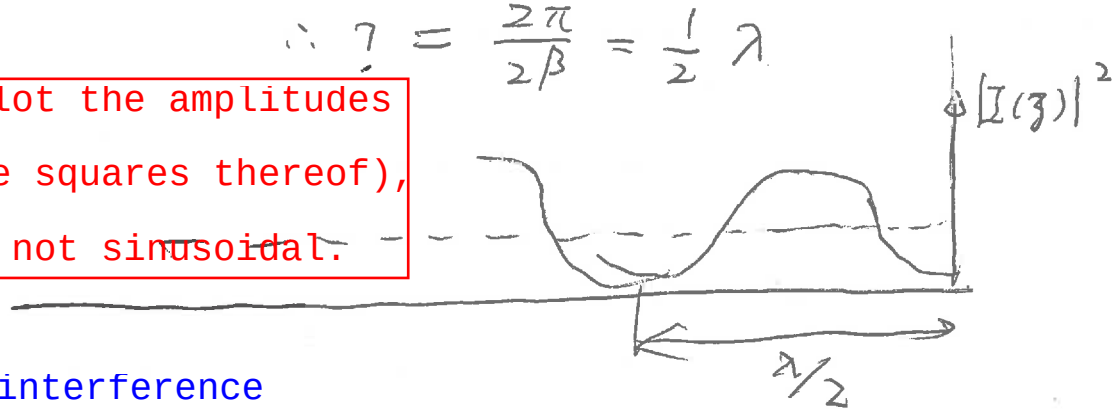


Pay attention to the max, min, and this value.

$$2\beta \lambda = 2\pi$$

$$\therefore \lambda = \frac{2\pi}{2\beta} = \frac{1}{2}\lambda$$

Note: If you plot the amplitudes (instead of the squares thereof), the curves are not sinusoidal.



constructive interference

$$|\tilde{V}|_{max} = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma|}$$

$$= |V_0^+| (1 + |\Gamma|)$$

destructive interference

$$|\tilde{V}|_{min} = |V_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma|}$$

$$= |V_0^+| (1 - |\Gamma|)$$

VSWR, or SWR

$$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Isn't this ~~obvious~~ ?!

When $|\Gamma| = 1$ $S = \infty$. all standing wave.

Sure, $|\tilde{V}|_{min} = 0$

When $|\Gamma| = 0$, $S = 1$. all traveling wave.
— no reflection

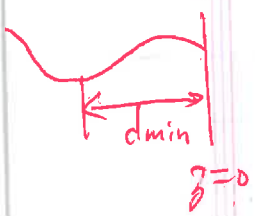
<u>Slotted line</u>

A tool to measure impedances.

Fig. 2-16, pp. 74 in 7/E

See the picture on p.p. 73 in 6/E.
(pp. 60 in 5/E).

sliding the detector, you find the maxima & minima, you get may not be the first one.



$S \left(= \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}}\right)$, & distance between mins. $= \lambda/2$

Solving $S = \frac{1+|\Gamma|}{1-|\Gamma|}$, you can find

$|\Gamma|$ not Γ yet !!!

The hope is that if you know Γ you can calculate Z_L by solving

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$ (you already know Z_0)

But $\Gamma = |\Gamma| e^{j\theta_r}$. We need to find θ_r

We know $\frac{\lambda}{2} \xrightarrow{\beta = \frac{2\pi}{\lambda}} \beta$

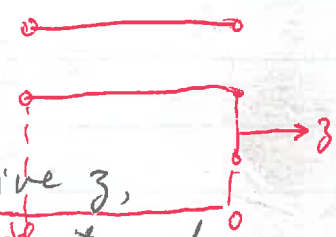
We know $z_{min} = -d_{min}$

$$2\beta z_{min} + \theta_r = -\pi \qquad z_{min} = -d_{min}$$

i.e $\quad -2\beta d_{min} + \theta_r = -\pi$

$$2\beta d_{min} - \theta_r = \pi \implies \theta_r$$

$$|\Gamma| e^{j\theta_r} = \Gamma \implies Z_L$$



We have so far always had negative z, because we draw the transmission line to the left of the load. We don't like to always carry the - sign.

So, we define $d = -z$.

$$\tilde{V}(z) = V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)$$

$$\implies V(d) = V_0^+ \left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)$$

Talk about sign & direction

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$$

$$\implies \tilde{I}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)$$

Let's now consider impedance looking into the transmission line @ d from the load.

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{\cancel{V_0^+}\left(e^{j\beta d} + \Gamma e^{-j\beta d}\right)}{\cancel{V_0^+}\left(e^{j\beta d} - \Gamma e^{-j\beta d}\right)} Z_0$$

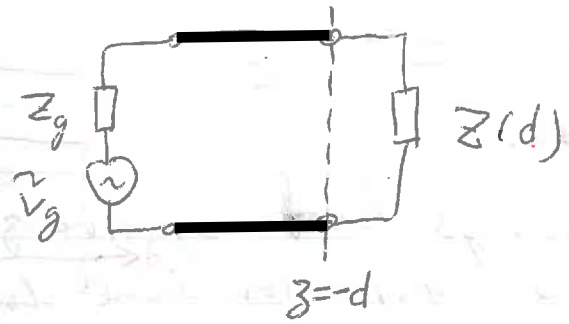
$$= Z_0 \cdot \frac{1 + \Gamma e^{-2j\beta d}}{1 - \Gamma e^{-2j\beta d}} \equiv Z_0 \cdot \frac{1 + \Gamma_d}{1 - \Gamma_d}$$

$$= \frac{\Gamma e^{-j\beta d}}{e^{j\beta d}}$$

$$\Gamma_d = \Gamma e^{-2j\beta d} = \frac{\text{Incident wave vol. amp. @ } d}{\text{Reflected " " " " "}}$$

complex

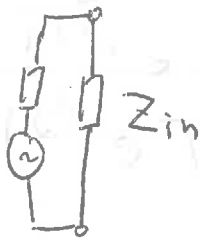
Equivalent ckt



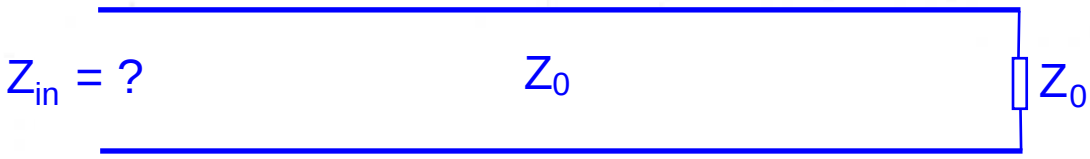
at the input end of the transmission line,

$$Z_{in} = Z(\underset{d=l}{l}) = Z_0 \cdot \frac{1+\Gamma_l}{1-\Gamma_l}$$

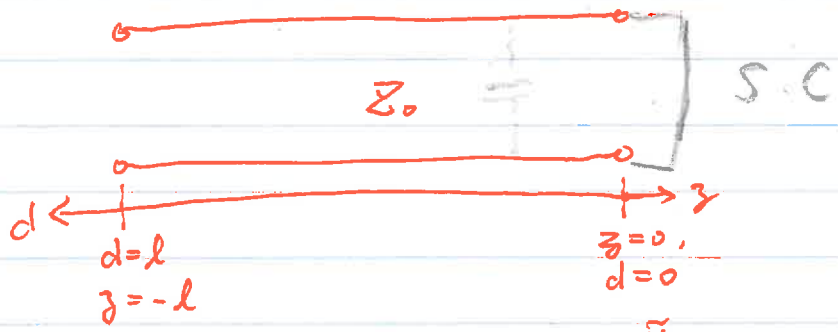
$\Gamma_l = \Gamma(l)$ @ the input.



Quiz:



Now, lets look at some special cases.
I mentioned a few times that a wire is may not be a good short for μwaves and open ends don't really mean open ckts.
But, for argument's sake, let's assume we can have a short ckt.



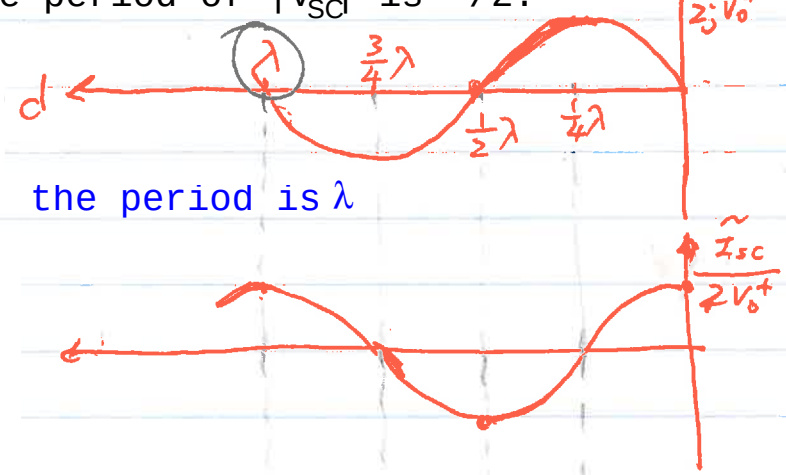
forward

$$\tilde{V}_{sc}(d) = V_0^+(e^{j\beta d} - e^{-j\beta d})$$
$$= 2jV_0^+ \sin\beta d$$

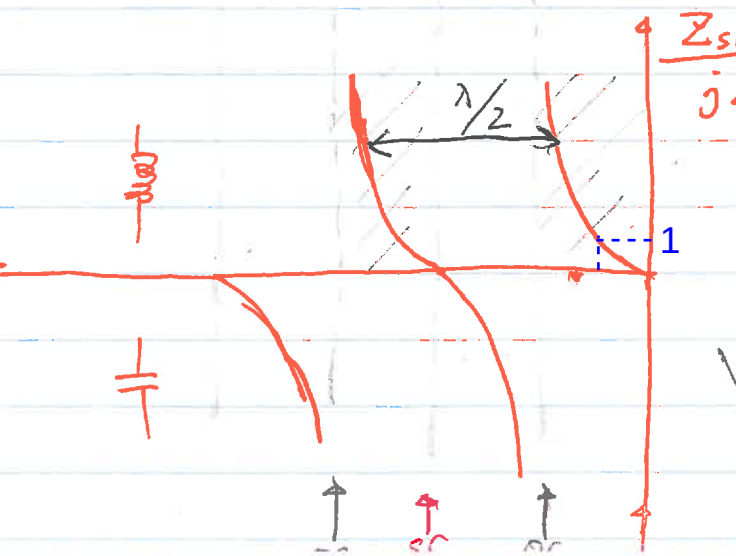
$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = \frac{2V_0^+}{Z_0}\cos\beta d$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)}$$
$$= jZ_0 \tan\beta d$$

The period of $|\tilde{V}_{sc}|$ is λ/2.



Here, the period is λ



The period of tan θ is π!

Now can you think of a way to make the transmission line feel like terminating w/ a s.c.?

The period for both the V & I standing waves is λ!

Think about this from a physics point of view.

Reactive loads don't consume power.
They take energy, and give back.
That's why you always have $|\Gamma| = 1$

Just a difference in phase.

short, ind. open, cap ...

Equivalent impedence

For $\tan \beta d > 0$

$$j\omega L_{eq} = j Z_0 \tan \beta d$$

$$L_{eq} = \frac{Z_0}{\omega} \tan \beta d$$

For $\tan \beta d < 0$

$$\frac{1}{j\omega C_{eq}} = j Z_0 \tan \beta d$$

$$C_{eq} = -\frac{1}{\omega Z_0 \tan \beta d}$$

Notice the frequency dependence.

Fig 2-21, pp. 82 (6/E)
(pp. 81, 7/E)

The open ckt case.

Just shift the d axis by $\frac{\lambda}{4}$

I'm not gonna repeat the stuff. Read on your own

With a short, you can make an open!

Now back to the general case

$$Z_{in} = Z(d=l) = Z_0 \cdot \frac{e^{j\beta l} + \Gamma e^{-j\beta l}}{e^{j\beta l} - \Gamma e^{-j\beta l}}$$

Insert the following:

at load.

$$\begin{cases} \Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{z_L - 1}{z_L + 1} \\ e^{j\beta l} = \cos\beta l + j\sin\beta l \\ e^{-j\beta l} = \cos\beta l - j\sin\beta l \end{cases}$$

normalized. Don't confuse w/

$$\boxed{z_L = \frac{Z_L}{Z_0} \quad \text{Normalized}}$$

$$\Rightarrow Z_{in} = \frac{Z_L \cos\beta l + j Z_0 \sin\beta l}{Z_0 \cos\beta l + j Z_L \sin\beta l} \cdot Z_0$$

$$z_{in} = \frac{z_L + j\sin\beta l}{1 + j z_L \sin\beta l}$$

~~If you like the normalized version. find it in the textbook.~~

The transmission line + load can be viewed as Z_{in}.

incident vs. input.

$$\tilde{V}_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} = V_0^+ (e^{j\beta l} + \Gamma e^{-j\beta l})$$

to be different from $\tilde{V}_{inc}(l) = V_0^+ e^{j\beta l}$ ☆

Solve this for V_0^+. you get:

$$\text{incident } V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

for a real problem just work back-words from the load.

$$Z_{in} = \frac{Z_L \cos\beta l + j Z_0 \sin\beta l}{Z_0 \cos\beta l + j Z_L \sin\beta l} \cdot Z_0$$

For $\beta l = n\pi$, i.e. $l = n \cdot \frac{\lambda}{2}$

$$Z_{in} = Z_L$$

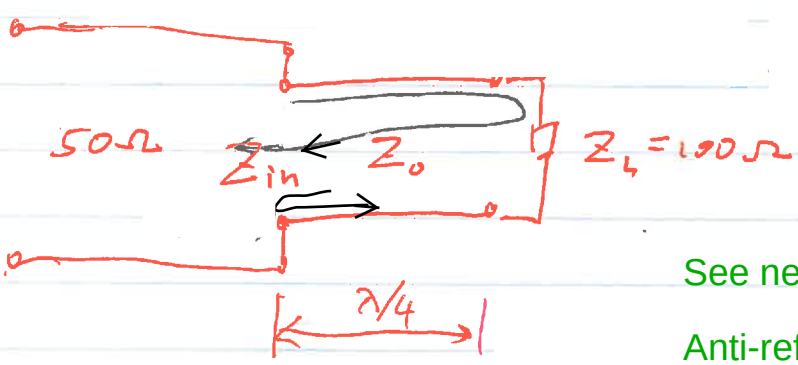
For $\begin{cases} \cos\beta l = 0 \\ \sin\beta l = 1 \end{cases}$, i.e. $\beta l = n\pi + \frac{1}{2}\pi$,

i.e. $l = n \cdot \frac{\lambda}{2} + \frac{\lambda}{4}$

$$Z_{in} = \frac{Z_0^2}{Z_L}$$

What's the equivalent normalized form?

$\boxed{\lambda/4 \text{ transformer}}$: one way of impedance matching.



$$Z_0 = \sqrt{Z_{in} \cdot Z_L}$$

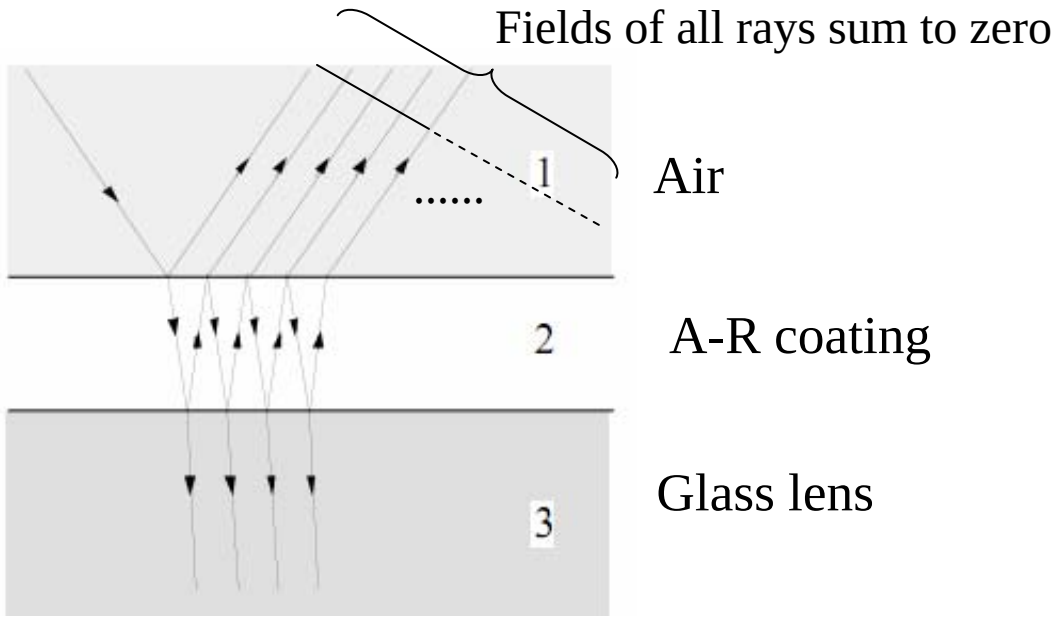
See next pages for optical analog:

Anti-reflective coating

Same principle as the optical anti-reflective coating

WITHOUT ANTI-REFLECTIVE WITH ANTI-REFLECTIVE

Anti-Reflection coating



Quarter wave transformer: multiple reflection view point

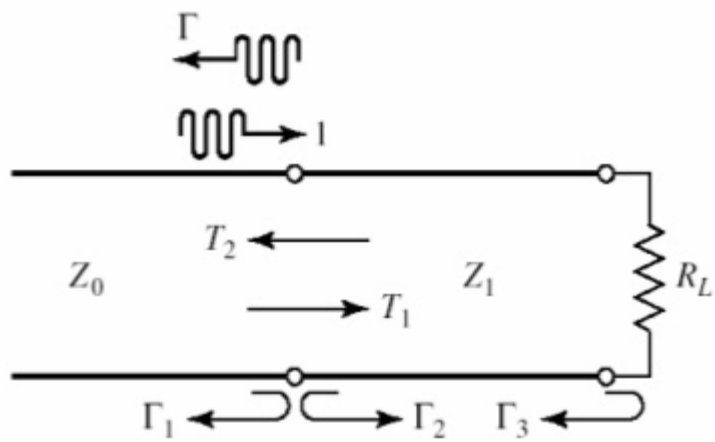
$$\Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o}$$

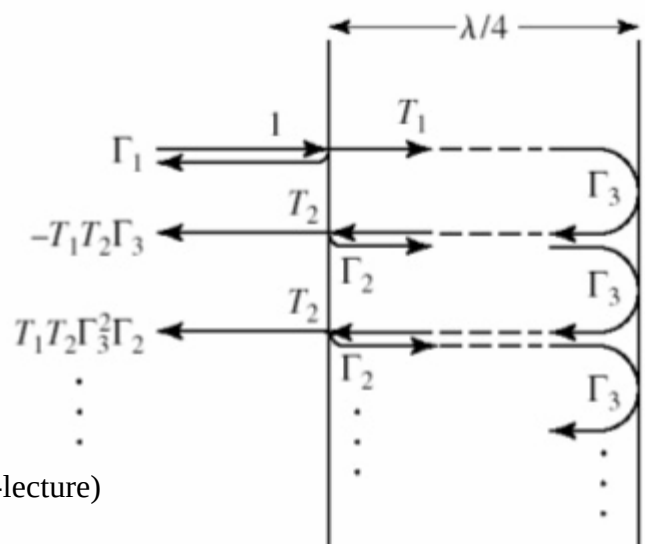
$$\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1}$$

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$

$$T_1 = \frac{2Z_1}{Z_1 + Z_o}$$

$$T_2 = \frac{2Z_o}{Z_1 + Z_o}$$





(Adapted from: Naveed Ramzan, http://www.slideshare.net/nramzan19/smith-chart-lecture)