we talked about the qualitative picture of standing waves. Here are the quantitative details.

Displacement of a string with one end fixed

\[ Y = Y_0^+ e^{-j\beta y} + Y_0^- e^{j\beta y} \]

\[ Y(0) = 0 \Rightarrow Y_0^+ + Y_0^- = 0 \]

\[ Y_0^- = -Y_0^+ \quad T = -1 \]

\[ Y = Y_0^+ (e^{-j\beta y} - e^{j\beta y}) \]

\[ e^{j\theta} - e^{-j\theta} = 2j\sin \theta \]

\[ Y = -2Y_0^+ j\sin (\beta z) \]  

The phasor is actually the complex amplitude.

\[ y(z, t) = 2Y_0^+ j\sin (\beta z) e^{j\omega t} \]

\[ \Rightarrow 2Y_0^+ \sin (\beta z) \sin \omega t \]

For simplicity, assume \( Y_0^+ \) is positive and real.

Quiz:

What property of a mirror makes it a mirror?

What's the current? Convert voltage and current phasors to time domain.

\[ T = -1, \quad V_0^- = -V_0^+ \]

\[ Z_l = 0 \quad S.C. \]

\[ V_0^+ = V_0^- \]

\[ I_0^+ = I_0^- \quad (\frac{Z_l}{Z_0} = 1, \quad T = +1) \]

\[ T = \frac{Z_l - Z_0}{Z_l + Z_0} = 1 \]

\[ Z_l = \infty \quad 0, \quad C \]

In practice, not easy to make an open circuit.

Phasor representation. You should know how to convert to time domain. (There was a homework problem)
Again, find the current. Convert both voltage and current phasors to time domain.

When \( \Gamma = \pm 1 \), completely standing wave.

Refer to Wiki, Standing wave.

At very high frequencies, we often can only measure the amplitude.

For short circuit, the amplitude of \( \tilde{V} \) is

\[
|\tilde{V}(z)| = \sqrt{\tilde{V}(z) \tilde{V}(\bar{z})} = \left| V_0^+ \sin \beta z \right| = 2 |V_0^+| \sin \beta z.
\]

Quiz:

What's the spatial period of the standing wave?

For short circuit:

\[
|\tilde{V}(z)| = 2 \left( V_0^+ \cos \beta z \right)
\]

For open circuit: \( |\tilde{V}(z)| = 2 \left( V_0^+ \cos \beta z \right) \)

\[
\tilde{V}(z) = 2 \left( 1 + \cos 2\beta z \right)
\]
\[ T = 1 \leq \frac{1}{T} \]

Read this, in general, it's complex. I'm confused with this combination of fractions and terms.

Now, let's look at the max and min of \( T \).

\[ |T| = 1 \]

What if \( |T| \neq 1 \)?

Wrong. \(|T| = 1\) means this is said to be completely stationary.

This is said to be completely stationary.

\[ 0 < |T| \] \( \min \)

\[ |T| = 2 \]

\[ |T| = 2 \] \( \max \)

\[ \text{explain} \]
\[ |\hat{V}(3)| = \sqrt{V(3) \hat{V}^*(3)} \]
\[ = \sqrt{V_0^+(e^{-j\beta_3} + |\Gamma| e^{j\theta_r} e^{j\beta_3})(V_0^+)^* e^{j\beta_3} + (|\Gamma| e^{j\theta_r} e^{j\beta_3})} \]

Notice that \( V_0^+(V_0^+)^* = |V_0^+|^2 \)
\[ \therefore |\hat{V}(3)| = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos (2\beta_3 + \theta_r)} \]
\[ = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos (2\beta_3 + \theta_r)} \]
\[ \text{interference term} \]

Similarly,
\[ |\hat{I}(3)| = |I_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma| \cos (2\beta_3 + \theta_r)} \]

\( \text{If not, just shift} \)
\[ \frac{2\pi}{2\beta} \cdot \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda} \]

\( \text{Note: If you plot the amplitudes (instead of the squares thereof), the curves are not sinusoidal.} \)

\[ |\hat{V}(3)|^2 \]
\[ = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos (2\beta_3 + \theta_r)} \]
\[ \text{constructive interference} \]
\[ |\hat{V}(3)|^2 \]
\[ = |V_0^+| (1 + |\Gamma|) \]
\[ \text{destructive interference} \]
\[ = |V_0^+| (1 - |\Gamma|) \]

Pay attention to the max, min, and this value.
VSWR, or SWR,
\[ S = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \]

When \(|\Gamma| = 1\), \(S = \infty\). All standing wave
Sure, \(|V_{\text{min}}| = 0\)

When \(|\Gamma| = 0\), \(S = 1\). All traveling wave.
--- no reflection

**Slotted Line.**
A tool to measure impedances.

See the picture on p.p. 73 in 6/E. (pp. 60 in 5/E).

Slide the detector, you find the maxima & minima, \(S = \frac{V_{\text{max}}}{V_{\text{min}}}, \text{ & distance between min.} = \lambda_0\)

Solving \(S = \frac{1 + |\Gamma|}{1 - |\Gamma|}\), you can find \(|\Gamma|\) not \(\Gamma\) yet!!!

The hope is that if you know \(\Gamma\), you can calculate \(Z_2\) by solving
\[ \Gamma = \frac{Z_2 - Z_0}{Z_2 + Z_0} \] (you already know \(Z_0\))

But \(\Gamma = |\Gamma| e^{i\theta}\). We need to find \(\theta\).
we know $\frac{\lambda}{2} \rightarrow \beta$

We know $z_{\text{min}} = -d_{\text{min}}$

$2\beta z_{\text{min}} + \theta_r = -\pi$

$i.e. -2\beta d_{\text{min}} + \theta_r = -\pi$

$2\beta d_{\text{min}} - \theta_r = \pi \Rightarrow \theta_r$

$|\Gamma| e^{j\theta_r} = 1 \Rightarrow Z_L$

We have so far always had negative $z$, because we draw the transmission line to the left of the load. $\circ$. We don't like to always carry the - sign.

So, we define $d = -z$

$\tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z})$

$\Rightarrow \tilde{V}(d) = V_0^+ (e^{j\beta d} + \Gamma e^{-j\beta d})$

$\tilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z})$

$\Rightarrow \tilde{I}(d) = \frac{V_0^+}{Z_0} (e^{j\beta d} - \Gamma e^{-j\beta d})$

Let's now consider impedance looking into the transmission line $d$ from the load.

$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+ (e^{j\beta d} + \Gamma e^{-j\beta d})}{V_0^+ (e^{j\beta d} - \Gamma e^{-j\beta d})} = Z_0 \cdot \frac{1 + \Gamma e^{-2j\beta d}}{1 - \Gamma e^{-2j\beta d}} = Z_0 \cdot \frac{1 + \Gamma d}{1 - \Gamma d}$
\[ \Gamma_d = \Gamma e^{-j \beta d} = \frac{\text{Incident wave vol. amp.}}{-j \beta d} \text{Reflected} \]

**Equivalent ckt**

At the input end of the transmission line,

\[ Z_{\text{in}} = Z(l) = Z_0 \cdot \frac{1 + \bar{\Gamma}_l}{1 - \bar{\Gamma}_l} \]

\[ \bar{\Gamma}_l = \Gamma(l) \oplus @ \text{the input} \]

**Quiz:**

\[ Z_{\text{in}} = ? \quad Z_0 \quad [Z_0] \]
Now, let's look some special cases.

I mentioned a few times that a wire is not a good short for waves, and open ends don't really mean open cts.

But, for argument's sake, let's assume we can have a short ckt.

\[ Z_0 = \]

\[ \tilde{V}_{sc}(d) = V_0^+ (e^{j\lambda d} - e^{-j\beta d}) \]

\[ = 2jV_0^+ \sin \beta d \]

\[ \tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0} (e^{j\beta d} + e^{-j\beta d}) \]

\[ = \frac{V_0^+}{Z_0} \frac{2 \sin \beta d}{2 \cos \beta d} = \frac{1}{Z_0} \tan \beta d \]

\[ Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} \]

\[ = \frac{1}{Z_0} \tan \beta d \]

The period of \( |\tilde{V}_{sc}| \) is \( \lambda/2 \).

Here, the period is \( \lambda \).

Now can you think of a way to make the transmission line feel like terra vis a sc.?
Think about this from a physics point of view.
Reactive loads don't consume power.
They take energy, and give back.
That's why you always have $|T| = 1$

Just a difference in phase.
short, ind, open, cap...

Equivalent impedance
For $\tan \beta d > 0$

$$jwL_{eq} = jZ_0 \tan \beta d$$

$$L_{eq} = \frac{Z_0}{w} \tan \beta d$$

For $\tan \beta d < 0$

$$\frac{1}{jwC_{eq}} = jZ_0 \tan \beta d$$

$$C_{eq} = -\frac{1}{wZ_0 \tan \beta d}$$

Notice the frequency dependence.

The open ckt case.

Just shift the d axis by $\frac{\pi}{4}$

I'm not gonna repeat the stuff. Read on your own.

With a short, you can make an open!
Now back to the general case

\[ Z_{in} = Z(d=l) = Z_0 \frac{e^{j\beta l} + Te^{-j\beta l}}{e^{j\beta l} - Te^{-j\beta l}} \]

Insert the following:

\[
T = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad \frac{Z_L}{Z_0} = \frac{3l-1}{3l+1} \quad \text{normalized}
\]

Don’t confuse us!

\[
e^{j\beta l} = \cos \beta l + j \sin \beta l \quad e^{-j\beta l} = \cos \beta l - j \sin \beta l
\]

\[ Z_{in} = \frac{Z_0 \cos \beta l + j Z_0 \sin \beta l}{Z_L \cos \beta l + j Z_L \sin \beta l} \quad Z_0
\]

\[ Z_{in} = \frac{\frac{Z_L}{3l+1} + j \frac{Z_L}{3l+1} \sin \beta l}{Z_L \cos \beta l + j Z_L \sin \beta l}
\]

If you like the normalized version. Find it in the textbook.

The transmission line + load can be viewed as \( Z_{in} \).

\[ V_i = \frac{V_o Z_{in}}{Z_0 + Z_{in}} = V_o^+ (e^{j\beta l} + T e^{-j\beta l}) \]

Incident vs. Input: \( V_{in} = V_o e^{j\beta l} \)

Solve this for \( V_o^+ \), you get:

\[ V_o^+ = \left( \frac{V_o Z_{in}}{Z_0 + Z_{in}} \right) \left( \frac{e^{j\beta l}}{e^{j\beta l} + T e^{-j\beta l}} \right) \]

For a real problem, just work backwards from the load.
\[ Z_{in} = \frac{Z_e \cos \beta l + j Z_0 \sin \beta l}{Z_0 \cos \beta l + j Z_e \sin \beta l} \cdot Z. \]

For \( \beta l = n \pi \), i.e. \( l = n \cdot \frac{\lambda}{2} \),

\[ Z_{in} = Z_z. \]

For \( \cos \beta l = 0 \), i.e. \( \beta l = n \pi + \frac{1}{2} \pi \),

\( \sin \beta l = 1 \), i.e. \( l = n \cdot \frac{\lambda}{2} + \frac{\lambda}{4} \).

\[ Z_{in} = \frac{Z_0^2}{Z_e} \]

What's the equivalent normalized form?

\( \frac{\lambda}{4} \text{ transformer} \) one way of impedance matching.

\[ Z_o = \sqrt{Z_{in} \cdot Z_e} \]

See next pages for optical analog:

Anti-reflective coating
Same principle as the optical anti-reflective coating
Anti-Reflection coating

Fields of all rays sum to zero

Air

A-R coating

Glass lens

Quarter wave transformer: multiple reflection view point

\[ \Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o} \]

\[ \Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} \]

\[ \Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1} \]

\[ T_1 = \frac{2Z_1}{Z_1 + Z_o} \]

\[ T_2 = \frac{2Z_o}{Z_1 + Z_o} \]

(Adapted from: Naveed Ramzan, http://www.slideshare.net/nramzan19/smith-chart-lecture)